CS 70 Fall 2003 — Discrete Mathematics for CS — Wagner — Final Soln

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your username on `cory.eecs`: ____

WRITE your section number (101 or 102): ___________________________

This exam is open-book, open-notes. *No calculators are permitted.* Do all your work on the pages of this examination. If you need more space, you may use the reverse side of the page, but try to use the reverse of the same page where the problem is stated.

You have 80 minutes. There are 4 questions, worth from 20 to 30 points each (100 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Do not turn this page until the instructor tells you to do so.

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

Problem 1. [Reasoning] (10 points)

(a) Prove or disprove: $((P \implies Q) \implies R) \implies ((R \implies Q) \implies P)$ is a tautology.

We construct a truth table. Let $S \equiv ((P \implies Q) \implies R) \implies ((R \implies Q) \implies P)$.

P	Q	R	$P \implies Q$	$(P \implies Q) \implies R$	$R \implies Q$	$(R \implies Q) \implies P$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F
F	T	F	T	F	T	F	T
F	F	T	T	T	F	T	T
F	F	F	T	F	T	F	T

Since S is false when $P = false$, $Q = R = true$, S is not a tautology.

(b) Let A, B be finite sets, with $|A| = m$ and $|B| = n$. How many distinct functions $f : A \to B$ are there from A to B?

For each element $a \in A$, we have n choices for the value of $f(a)$. Thus there are n^m possible functions.

Problem 2. [Induction] (15 points)

Prove by induction: $5x \le x^2 + 6$ for all $x \in \mathbf{N}$.

Let $P(x) =$ "$5x \le x^2 + 6$." We prove that $\forall x \in \mathbf{N}.\ P(x)$.

Base cases: $x = 0$ and $x = 1$.
We have $0 \le 6$ for $x = 0$, and $5 \le 7$ for $x = 1$.

Inductive step: $P(x) \implies P(x+1)$.
We have

$$\begin{aligned} 5x &\le x^2 + 6 && \text{(by the inductive hypothesis)} \\ 5x + 5 &\le x^2 + 11 \\ 5(x+1) &\le (x^2 + 2x + 1) - 2x + 10 \\ &\le (x+1)^2 + 6 - (2x - 4) \\ &\le (x+1)^2 + 6. && \text{(since } 2x \ge 4 \text{ for } x \ge 2) \end{aligned}$$

Thus by the principle of induction, $5x \le x^2 + 6$ for all $x \in \mathbf{N}$.

Problem 3. [Modular Arithmetic] (20 points)

(a) Let X be uniformly distributed on $\{0, 1, \ldots, 16\}$. What is $\Pr[2X + 7 \equiv 0 \pmod{17}]$?

2 is invertible mod 17, so the only solution is $X = 5$. Thus $\Pr[2X + 7 \equiv 0 \pmod{17}] = \frac{1}{17}$.

(b) Let X be uniformly distributed on $\{0, 1, \ldots, 32\}$. What is $\Pr[2X + 7 \equiv 0 \pmod{33}]$?

2 is invertible mod 33, so the only solution is $X = 13$. Thus $\Pr[2X + 7 \equiv 0 \pmod{33}] = \frac{1}{33}$.

(c) Let X be uniformly distributed on $\{0, 1, \ldots, 32\}$. What is $\Pr[3X + 12 \equiv 0 \pmod{33}]$?

3 is not invertible mod 33, so there are multiple solutions: $X = 7$, $X = 18$, and $X = 29$. You can see that these are the only solutions by noting that any solution must satisfy $3X + 1 \equiv 0 \pmod{11}$, i.e., $X \equiv 7 \pmod{11}$. Thus $\Pr[3X + 12 \equiv 0 \pmod{33}] = \frac{1}{11}$.

(d) Let X be uniformly distributed on $\{0, 1, \ldots, 40\}$. What is $\Pr[X^2 + 40 \equiv 0 \pmod{41}]$?

Rearranging the congruency, we have $X^2 \equiv -40 \pmod{41}$, or $X^2 \equiv 1 \pmod{41}$. The two obvious solutions are $X \equiv \pm 1 \pmod{41}$. There cannot be any more solutions, because 41 is prime, and any polynomial of degree 2 has at most 2 roots modulo a prime. Thus $\Pr[X^2 + 40 \equiv 0 \pmod{41}] = \frac{2}{41}$.

Problem 4. [Probability] (20 points)

For each square of a 8×8 checkerboard, flip a fair coin, and color that square black or red according to whether you get heads or tails. Assume that all coin flips are independent.

A *same-color row* is a row on the board where all squares in the row have the same color (i.e., all red, or all black). Let the random variable X denote the number of same-color rows.

(a) If all coin tosses come up heads, what is the value of X?

The value of X is the number of rows, so $X = 8$.

(b) Calculate $\Pr[X = 0]$. (You do not need to simplify your answer.)

Let X_i be an indicator variable that is 0 if the ith row is not all of the same color, and 1 if the ith row is all of the same color. Then $X_i = 1$ only when the entire row is either heads or tails, so $\Pr[X_i = 1] = \frac{2}{2^8} = \frac{1}{2^7}$. Then $\Pr[X_i = 0] = 1 - \frac{1}{2^7}$. Now notice that $\Pr[X = 0] = \bigcap_i \Pr[X_i = 0]$, and that these events are mutually independent. Thus $\Pr[X = 0] = \prod_i \Pr[X_i = 0] = (1 - \frac{1}{2^7})^8 \approx 0.94$.

(c) Calculate $\mathbf{E}[X]$.

We calculate $\mathbf{E}[X_i] = \frac{1}{2^7}$. Then by linearity of expectation $\mathbf{E}[X] = \sum_i \mathbf{E}[X_i] = 8 \times \frac{1}{2^7} = \frac{1}{2^4} = \frac{1}{16}$.

(d) Calculate $\mathrm{Var}[X]$.

We have $\mathbf{E}[X_i^2] = \frac{1}{2^7}$. Then $\mathrm{Var}[X_i] = \frac{1}{2^7} - (\frac{1}{2^7})^2 = \frac{127}{16384} \approx 0.0078$. Now the X_i's are independent, so $\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] = \frac{127}{2048} \approx 0.062$.

(e) Show that $\Pr[X \geq 3] \leq 1/48$.

From Markov's inequality, we have $\Pr[X \geq 3] \leq \frac{1}{3}\mathbf{E}[X] = \frac{1}{48}$.

Problem 5. [Probability] (20 points)

A gambler has 4 coins in her pocket. Two are double-headed, one is double-tailed, and one is normal. The coins cannot be distinguished unless one looks at them.

(a) The gambler shuts her eyes, chooses a coin at random, and tosses it. What is the probability that the lower face of the coin is heads?

There are 5 faces that are heads out of a total of 8, so the probability is $\frac{5}{8}$.

(b) She opens her eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head?

Let A be the event that the upper face is a head, and B be the event that the lower face is heads. From (a), $\Pr[A] = \Pr[B] = \frac{5}{8}$. Now $\Pr[A \cap B] = \frac{1}{2}$, so $\Pr[B|A] = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}$.

(c) Now, after having seen that the upper face is a head, she shuts her eyes again, picks up the same coin, and tosses it a second time. What is the probability that the lower face is a head?

From (b), the probability that both faces are heads is $\frac{4}{5}$, in which case the lower face is guaranteed to be a head, and the probability that only one face is a head is $\frac{1}{5}$, in which case the lower face is a head with probability $\frac{1}{2}$. Thus the total probability that the lower face is a head is $\frac{4}{5} \times 1 + \frac{1}{5} \times \frac{1}{2} = \frac{9}{10}$.

(d) After her second toss (as described in part (c)), she opens her eyes and sees that the upper face is a head. What is the probability that the lower face is a head?

Let C be the event that the upper face is a head, and D be the event that the lower face is heads. From (c), $\Pr[C] = \Pr[D] = \frac{9}{10}$. Now $\Pr[C \cap D] = \frac{4}{5}$, so $\Pr[D|C] = \frac{\frac{4}{5}}{\frac{9}{10}} = \frac{8}{9}$.

Problem 6. [Probability] (15 points)

Choose a number uniformly at random between 0 and 999,999, inclusive. What is the probability that the digits sum to 19?

Let x_1, x_2, $\cdots$, x_6 be variables corresponding to each digit in a number. Then we calculate the number of ways to assign these variables such that they sum to 19, i.e. the number of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19, \qquad x_1 \geq 0, \ldots, x_6 \geq 0.$$

Using stars and bars, the number of solutions is $\binom{24}{5}$. But this also counts the number of solutions for which some x_i is 10 or higher, and we can't have a digit value greater than 9. Thus we need to subtract the number of solutions for which some x_i is ≥ 10. Notice that, because $19 < 10 + 10$, we cannot have two such x_i's, so we just need to count the solutions $(x_1, \ldots, x_6)$ where exactly one of the x_i's is 10 or higher. Subtracting 10 from such an x_i and from the total sum, we obtain the equation

$$x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 9, \qquad x'_1 \geq 0, \ldots, x'_6 \geq 0.$$

There are $\binom{14}{5}$ solutions to this equation. But since there are six choices for which variable is greater than 10, the total amount we overcounted by is $6 \cdot \binom{14}{5}$. Thus the total number of six digit numbers whose digits sum to 19 is $\binom{24}{5} - 6 \cdot \binom{14}{5}$. The probability that a random six digit number has digits that sum to 19 then is $\frac{\binom{24}{5} - 6 \cdot \binom{14}{5}}{1{,}000{,}000} = \frac{42504 - 6 \times 2002}{1{,}000{,}000} = \frac{30492}{1{,}000{,}000} = 0.030492$.

Thanks for a great semester! Have a refreshing holiday break,
. — David & Amir