CS 70 Fall 2000 — Discrete Mathematics for CS — Papadimitriou/Russell/Sinclair — Midterm 1

You have two hours. The exam is open-book, open-notes.
There are five *questions, 100 points total*

You should be able to finish all the questions, so avoid spending too long on any one question. Write your answers in blue books. Check you haven't skipped any by accident. Hand them all in. Panic not.

1. (16 pts.) Rational numbers

(a) (4 pts) Prove that if a and b are rational, then ab is rational.

(b) (6 pts) Prove that if x is irrational, $x^{\frac{1}{3}}$ is irrational.

(c) (6 pts) State the *converse* of (b). State whether the converse is true or false and prove your assertion.

2. (20 pts.) Induction

(a) (10 pts) Prove by induction that $5^n - 1$ is divisible by 4 for all $n \geq 1$.

(b) (10 pts) Let $S(n) = \sum_{i=1}^{n} i$ denote the sum of the first n natural numbers. What is wrong with the following "proof" that $S(n) = \frac{1}{2}(n + \frac{1}{2})^2$ for all $n \geq 1$?

The claim is obviously true for the base case, $n = 1$. Now consider the inductive step. We have

$$\begin{aligned} S(n+1) &= S(n) + (n+1) \\ &= \tfrac{1}{2}(n+\tfrac{1}{2})^2 + (n+1) \qquad \text{by inductive hypothesis} \\ &= \tfrac{1}{2}(n^2 + n + \tfrac{1}{4}) + n + 1 \\ &= \tfrac{1}{2}(n^2 + 3n + \tfrac{9}{4}) \\ &= \tfrac{1}{2}((n+1) + \tfrac{1}{2})^2 \end{aligned}$$

This completes the proof by induction.

3. (18 pts.) Satisfiability and all that

For each of the following Boolean expressions, decide if it is (i) valid; or (ii) satisfiable but not valid; or (iii) unsatisfiable. Justify all your answers.

(a) (6 pts) $(A \vee B) \wedge (B \vee C) \wedge (C \vee A)$;

(b) (6 pts) $(A \wedge \neg B) \vee (B \wedge \neg C) \vee (C \wedge \neg A)$;

(c) (6pts) $A \wedge (\neg A \vee B) \wedge (\neg A \vee \neg B)$.

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4. (24 pts.) Recursion

The following algorithm, given as inputs a real number x and a natural number y, is supposed to output the value x^y.

algorithm $\exp(x, y)$
  if $y = 0$ then return(1)
  else
    $z := \exp(x, \lfloor \frac{y}{2} \rfloor)$
    if y is even then return($z \times z$)
    else return($x \times z \times z$)

(a) (4 pts) Hand-turn the algorithm to compute $\exp(3, 7)$. Show your working.

(b) (10 pts) Prove by induction on y that the algorithm is correct.

(c) (10 pts) Prove, also by induction, that the number of multiplications performed by the algorithm is no more than $2b(y)$, where $b(y)$ is the number of bits in the binary representation of y.

5. (22 pts.) Kowalski Normal Form

A Boolean expression is in *Kowalski Normal Form* (KNF) if it is a conjunction of implications, where the premise of each implication is a conjunction of variables and the conclusion of each implication is a disjunction of variables. (Note: *variables*, not *literals*.) For example, the expression $((B \wedge C) \implies (A \vee D)) \wedge (C \implies E)$ is in KNF.

(a) (6 pts) Prove that $(B \wedge C) \implies (A \vee D)$ is logically equivalent to $(A \vee \neg B \vee \neg C \vee D)$. You may use any logical equivalence or other theorem stated in the lecture notes.

(b) (10 pts) Prove by induction the generalized form of de Morgan's law, which states that

$$\neg(X_1 \vee \cdots \vee X_n) \equiv (\neg X_1 \wedge \cdots \wedge \neg X_n)$$

(c) (6 pts) Now use part (b) to prove that every Boolean expression is logically equivalent to an expression in KNF.