CS 70 Spring 2000 — Discrete Mathematics for CS — Wagner — Midterm 1

You have 1 hour and 20 minutes. The exam is open-book, open-notes.
100 points total (five questions of varying credit as shown).

You should be able to finish all the questions, so avoid spending too long on any one question..
Write your answers in blue books. Check you haven't skipped any by accident. Hand them all in. Panic not.

1. (20 pts.) Satisfiability and all that

For each of the following Boolean expressions, decide if it is (i) valid (ii) satisfiable (iii) unsatisfiable. (Give *all* applicable properties, with justifications.)

(a) (5) $A \wedge \neg A \wedge \neg B$

(b) (10) $(A \implies B) \wedge (B \implies C) \wedge (C \implies \neg A)$

(c) (5) $(A \implies B) \vee (B \implies A)$

2. (25 pts.) Induction

Given a binary tree t, the depth $depth(l,t)$ of a leaf l in t is defined to be the length of the path from l to the root of t. (Hence, if t is just an atom l, then $depth(l,t) = 0$.)

(a) (3) In the tree t below, what are the depths of the 3 leaves A, B, C of t?

(b) (4) If a leaf l is a leaf of tree t_1 and $depth(l,t_1) = d$, and t_2 is some other tree, what is the value of $depth(l, t_1 \bullet t_2)$? (You do not have to *prove* your answer correct.)

(c) (3) Let us define

$$L(t) = \sum_{l\,:\,l \text{ is a leaf of } t} \frac{1}{2^{depth(l,t)}}.$$

Verify that $L(t) = 1$ for the tree above.

(d) (15) Using tree induction, show that $L(t) = 1$ for every binary tree t. (You may use the result of part (2b) as a lemma here without giving the proof.)

3. (15 pts.) CNF and validity

(a) (5) Consider a clause $C \equiv (L_1 \vee \ldots \vee L_k)$, where the L_i's are literals. Show that C is valid if and only if, for some $i, j \in \{1, \ldots, k\}$, $L_i \equiv \neg L_j$.

(b) (5) Use the result from part (a) to design (and justify) an efficient algorithm that checks the validity of an input formula given in CNF.

(c) (5) Convert the expression $((A \vee B) \implies C) \implies (A \implies C))$ directly into CNF using standard logical equivalences (do not simplify along the way). Hence check its validity using your algorithm of part (b).

4. (15 pts.) Coloring a Map

(a) (10) A *map* is a set of n countries $C_1,\ldots C_n$, plus a specification of which countries C_i are adjacent to which countries C_j. A *feasible 2-coloring* assigns one of two colors to each country, such that no adjacent countries are the same color. (For example, the squares of a chessboard have a feasible 2-coloring.)

Given a map, explain how to construct a CNF expression that is satisfiable iff a feasible 2-coloring exists for the map.

(b) (5) Explain how to use a CNF satisfiability-checker to prove that two given countries (call them C_1 and C_2) must be the same color in any feasible 2-coloring of a given map.

5. (25 pts.) Proofs

(a) (10) Consider the following:

Theorem 0.1: *For all integers* $n \geq 1$, *we have* $5n-5=0$.

Plainly this "theorem" is false. What is wrong with the following "proof"?

Proof: We use strong induction on **N**.

- Base case ($n=1$): $5\cdot 1-5=0$.
- Inductive step:

$$\begin{aligned} 5(n+1)-5 &= 2(5n-5)-(5(n-1)-5) \\ &= 2(0)-0 \\ &= 0. \end{aligned}$$

□

(b) (15) The *harmonic numbers* H_k, k = 1, 2, 3, ..., are defined by

$$H_k = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} = \sum_{i=1}^{k}\frac{1}{i}.$$

We propose to prove the following Theorem:

Theorem 0.2: *For all* $n \geq 0$, $H_{2^n} \geq 1+n$.

In our proof, we will use (without proof) the following easy lemma:

Lemma. For each $n \geq 0$, we have $\sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i} \leq 1$.

What is wrong with the following "proof" of the Theorem? [Hint: There is nothing wrong with the Lemma above.]

Proof: We use simple induction on **N**:

- Base Case ($n=0$): $H_{2^0} = 1 \geq 1+0$
- Inductive step:

$$\begin{aligned} H_{2^{n+1}} &\geq 1+(n+1) \\ \sum_{i=1}^{2^{n+1}}\frac{1}{i} &\geq 1+n+1 \\ \sum_{i=1}^{2^n}\frac{1}{i}+\sum_{i=2^n+1}^{2^{n+1}}\frac{1}{i} &\geq 1+n+1 \\ \sum_{i=1}^{2^n}\frac{1}{i} &\geq 1+n \end{aligned}$$

(where the last step in the derivation used the Lemma, and the fact that $A \geq B, C \geq D$ gives $A+C \geq B+D$). QED.

□