

# A Simple Provable Algorithm for Curve Reconstruction

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## Abstract

We present an algorithm that provably reconstructs a curve in the framework introduced by Amenta, Bern and Eppstein. The highlights of the algorithm are: (i) it is simple, (ii) it requires a sampling density better than previously known, (iii) it can be adapted for curve reconstruction in higher dimensions straightforwardly.

## 1 Introduction

We consider the problem of curve reconstruction that takes a set of sample points on a smooth closed curve  $C$ , and requires to produce a geometric graph  $G$  having exactly those edges that connect sample points adjacent in  $C$ . Obviously, given only the samples, it is not always possible to compute  $G$  unless some additional conditions are satisfied by the input. Amenta, Bern and Eppstein [1] proposed a framework based on *local feature size* under which they show two graphs, *crust* and  $\beta$ -*skeleton*, coincide with  $G$  if the points are sufficiently sampled. Some of the other effective approaches include  $\alpha$ -shapes by [6] which is analyzed later by [3],  $r$ -regular shapes by [2],  $\mathcal{A}$ -shapes by [7] and a Delaunay based method by [4]. A survey of these methods appear in [5]. In this paper we show that a modified nearest neighbor graph also coincides with  $G$ . The algorithm and its analysis are simple. Nevertheless, it improves the sampling density to  $1/3$  from  $0.252$  as required by [1]. More importantly, the algorithm generalizes to higher dimensional curve reconstruction almost straightforwardly. It is not hard to verify that all lemmas and theorem of section 3 hold in any ambient Euclidean space.

We require the following definitions most of which have been introduced in [1]. The *medial axis*  $M$  of a smooth curve  $C$  in  $R^d$  is the closure of all points that have two or more closest points in  $C$ . The *local feature size*  $f(p)$  at a point  $p \in C$  is the least Euclidean distance of  $p$  from  $M$ . A point set  $P \subseteq C$  is an  $\epsilon$ -sample of  $C$  if and only if each point  $p \in C$  has a sample within  $\epsilon f(p)$  distance. The angle between two edges sharing a common point is the smaller of the two planar angles made by them. We denote the Euclidean

distance between two points  $p, q$  and the length of an edge  $e$  with  $\ell(pq)$  and  $\ell(e)$  respectively.

## 2 The algorithm

Algorithm NN-CRUST(input: an  $\epsilon$ -sample  $P$ )

Step 1: Compute the set of edges  $N$  that connect nearest neighbors in  $P$ .

Step 2: Let  $a$  be a point that is incident with only one edge  $e$  in  $N$ . Compute the shortest edge incident with  $a$  among all the edges that make an angle more than  $\pi/2$  with  $e$ . Let  $D$  be the set of all such edges.

Step 3: Output  $G = N \cup D$ .

Both steps 1 and 2 can be performed on the edges of the Delaunay triangulation  $T$  of  $P$  since the desired graph  $G$  is known to be contained in  $T$  [1]. This implies that, in  $R^2$ , all steps of NN-CRUST takes time  $O(n)$  once  $T$  is computed in time  $O(n \log n)$ , where  $n$  is the number of points in  $P$ .

## 3 Proof of correctness

The first lemma is easily deducible from triangular inequality, the second one is proved in [1], and we skip the proof of the third one.

LEMMA 3.1.  $f(q) \leq f(p) + \ell(pq)$  for any two points  $p, q$  in  $C$ .

LEMMA 3.2. If  $B$  is a closed ball with  $B \cap C$  not a 1-disk, then  $B$  contains a medial axis point.

LEMMA 3.3. The angle between two adjacent edges in  $G$  is more than  $\pi/2$  if  $\epsilon \leq 1/3$ .

LEMMA 3.4.  $\ell(e) < \frac{2\epsilon}{1-\epsilon} f(p)$  for any edge  $e \in G$ , where  $p$  is an endpoint of  $e$  and  $\epsilon < 1$ .

*Proof:* Let  $q$  be the point where the perpendicular bisector of  $e = ab$  intersects the portion of  $C$  over which  $a$  and  $b$  are adjacent. Grow a ball centered at  $q$  until it touches the two endpoints of  $e$ . The growing ball always intersects  $C$  in a 1-disk since otherwise its radius would be greater than or equal to  $f(q)$  (Lemma 3.2) when it had touched the first sample; a case eliminated by the sampling condition at  $q$  with  $\epsilon < 1$ . It follows that the

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two endpoints of  $e$  are the nearest samples to  $q$ . This implies  $\ell(e) \leq 2\epsilon f(q)$ . Substitute  $f(q)$  by  $\frac{\epsilon}{1-\epsilon}f(p)$  since Lemma 3.1 gives  $f(q) \leq f(p) + \ell(pq) \leq f(p) + \epsilon f(q)$ .

LEMMA 3.5. *Let  $e \notin G$  be any edge between two samples and  $a$  be any of its endpoints. Then, either  $\ell(e) > f(a)$ , or there is an edge  $h \in G$  incident with  $a$  which makes an angle less than  $\pi/2$  with  $e$  and  $\ell(h) < \ell(e)$ .*

*Proof:* Consider the closed ball  $B$  with  $e$  as diameter. In case  $C_e = B \cap C$  is a 1-disk, there must be an edge  $ax \in G$  where  $x$  lies in  $C_e$ . Otherwise,  $e \in G$ . It follows that the edge  $ax$  sharing an endpoint  $a$  with the diameter  $e$  must make an angle less than  $\pi/2$  with it and  $\ell(ax) < \ell(e)$ .

In the other case when  $C_e$  is not a 1-disk, apply Lemma 3.2 to conclude that  $B$  has a medial axis point and hence  $\ell(e) > f(a)$ .

LEMMA 3.6. *Let  $a$  be any sample and  $b$  its nearest neighbor. The edge  $ab$  is in  $G$  if  $\epsilon \leq 1/3$ .*

*Proof:* Suppose, on the contrary,  $ab \notin G$ . Then, we argue that both conditions of Lemma 3.5 are violated reaching a contradiction. Let  $ax$  be an edge in  $G$ . First consider the case of  $\ell(ab) > f(a)$ . With  $\epsilon \leq 1/3$  we have  $\ell(ax) < \frac{2\epsilon}{1-\epsilon}f(a) \leq f(a)$  (Lemma 3.4). This gives  $\ell(ax) < \ell(ab)$ , an impossibility since  $b$  is the nearest neighbor to  $a$ . Next, consider the case  $\ell(ab) \leq f(a)$ . According to Lemma 3.5 there is an edge  $ax$  in  $G$  so that  $\ell(ax) < \ell(ab)$  reaching a contradiction.

THEOREM 3.1. *Given an  $\epsilon$ -sample for a closed curve with  $\epsilon \leq 1/3$ , the algorithm NN-CRUST outputs an edge  $e$  if and only if  $e \in G$ .*

*Proof:* Let  $e = ab$  be an edge computed by the algorithm. Let  $ax, ay$  denote the two edges in  $G$  that are incident with  $a$ . If  $e$  is computed in step 1, it is in  $G$  due to Lemma 3.6. Otherwise, it is computed in step 2 which means one of the edges  $ax$  and  $ay$ , say  $ax$ , has already been computed in step 1. The edge  $e$  makes an angle more than  $\pi/2$  with  $ax$ . The edge  $ay$  also makes an angle more than  $\pi/2$  with  $ax$  due to Lemma 3.3. If  $e \notin G$ , then Lemma 3.5 applies to conclude that  $\ell(ay) < \ell(e)$ . But, that is impossible since the algorithm chose  $e$  to be the shortest edge making angle more than  $\pi/2$  with  $ax$ .

To show the the other direction consider any edge  $e = ab$  in  $G$ . If  $e$  is a nearest neighbor edge then it is computed in step 1. Otherwise, the other edge in  $G$  incident with  $a$ , say  $ax$ , must be a nearest neighbor edge and has been computed in step 1. The edge  $e$  makes an angle more than  $\pi/2$  with  $ax$  and  $e$  is the shortest among

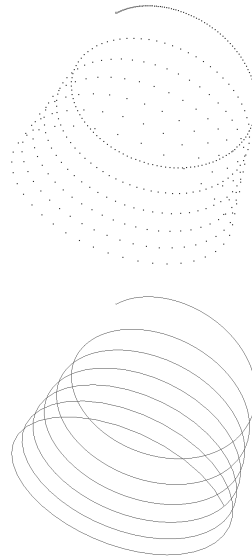


Figure 1: A reconstructed curve in 3D

all such edges. Otherwise, Lemma 3.5 is violated. This means that  $e$  is computed in step 2.

**An example:** In Figure 1 we show a reconstruction in 3D. The 550 points are sampled from the parametric curve  $x = \sin t^2$ ,  $y = \cos t^2$ ,  $z = t/3.0$ . This is a case of a curve with endpoints. We took care of the endpoints specially in the program.

## References

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