Satisfiability For Arithmetic Using Simplex

Have a set of linear inequalities

$$a_1' + \bar{a}_1 \cdot \bar{x} \geq 0$$
$$\vdots$$
$$a_i' + \bar{a}_i \cdot \bar{x} \geq 0$$

Equalities, and strict inequalities can be converted to this form for integer variables and coefficients

Simplex is a linear programming algorithm
- can be used to maximize (minimize) a linear expression subject to a set of linear inequality constraints
- this is useful for satisfiability

To find if $S \Rightarrow a' + \bar{a}\bar{x} \not\leq 0$

first maximize $a' + \bar{a} \cdot \bar{x}$ to k over S

<u>Means</u>: $S \Rightarrow a' + \bar{a} \cdot \bar{x} \leq k$ and

$S \wedge a' + \bar{a} \cdot \bar{x}$ is satisfiable

Then $S \Rightarrow a' + \bar{a} \cdot \bar{x} \not\leq 0$ iff $k \not\leq 0$

- But we need more — Simplex will offer
  - undo support ----> inexpensive
  - generate equalities ----> simple
  - generate proofs ----> simple

Representing the inequalities in the Simplex tableau

- for each inequality $a' + \bar{a} \cdot \bar{x} \geqslant 0$ we introduce a slack variable s and two constraints

$$s = a' + \bar{a} \cdot \bar{x}$$

$$s \geqslant 0$$

- we have equality constraints and $s \geqslant 0$ inequality constraints
- we represent them in a Simplex tableau

		C_1	$\dots$	C_j	$\dots$	C_c
R_1	T_{10}	T_{11}	$\dots$	T_{1j}	$\dots$	T_{cj}
$\vdots$						
R_i	T_{i0}		$\dots$	T_{ij}	$\dots$	T_{ic}
R_r	T_{r0}		$\dots$	T_{rj}	$\dots$	T_{rc}

- R_i are the expressions that own row i
- C_j – " – column j
  - they operate as slack variables

Invariant 1

$$R_i \simeq T_{i0} + \sum_j' T_{ij} \cdot C_j$$

where $E_1 \simeq E_2$ means that

$E_1 - E_2$ can be simplified to 0 using the axioms of $(\mathbb{Z}, 0, +)$ as a commutative group.

E.g.

$$2x + y \simeq 2(x + 2\cdot(y - z)) + 2z - 3y$$

- So far we represented only the equality constraints
- To represent the ≥ 0 constraints we mark the owner of a row or a column as "+-restricted" row or column

Invariant 2

- If row i is +-restricted then there is a proof of $R_i \geq 0$. Call that Proof(R(i))
- Similar for +-restricted columns

Example

$x+1 \geq y$ $\qquad$ $x+y \geq -3$ $\qquad$ $-x \geq 4$

		x	y
a^*	1	1	-1
b^+	3	1	1
c^+	-4	-1	0

$C_1 = x \qquad C_2 = y$

$R_1 = x - y + 1$

$R_2 = x + y + 3$

$R_3 = -x - 4$

But these inequations are not satisfiable

$$(x+1 \geq y) + (x+y \geq -3) + 2 \cdot (-x \geq 4)$$
$$=$$
$$1 \geq -3 + 8$$

Idea: a tableau can become unsatisfiable only when restricting rows or columns

- convenient to work with satisfiable tableaux
- only restrict the tableau if we can verify that it stays satisfiable

Definition

- The sample point of the tableau is obtained by setting all C_j to 0 and all $R_i = T_{i0}$
  - this satisfies all equality constraints
- A tableau is feasible if the sample points satisfies all row restrictions

$$\boxed{\text{for all +-restr. row } i \quad T_{i0} \geq 0}$$

Back to our example

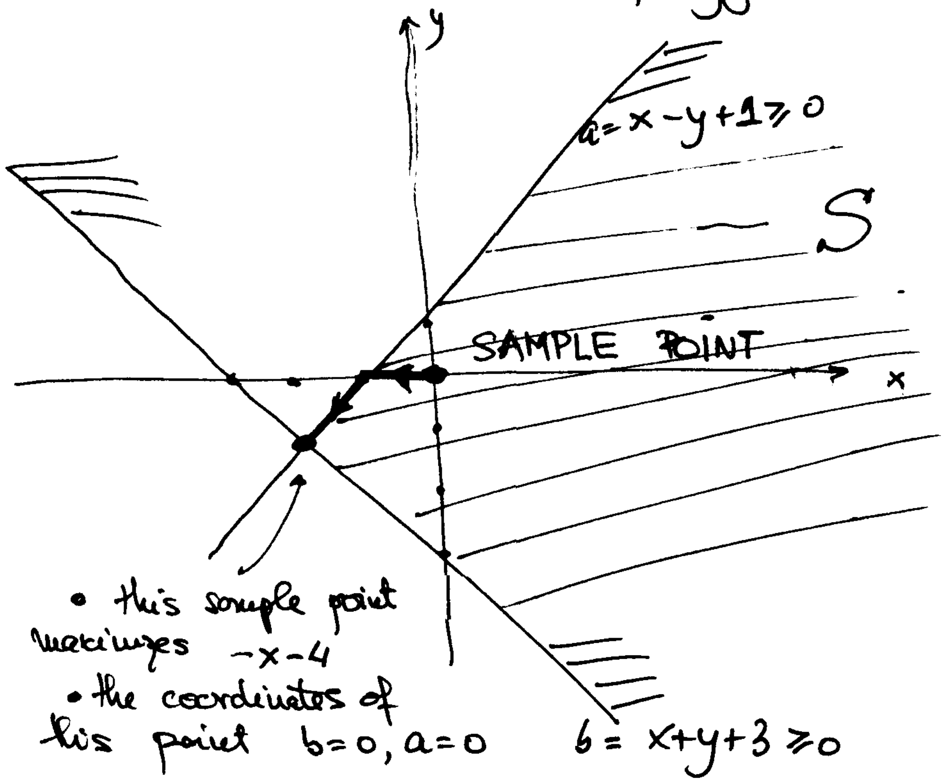
- we hold on to the restriction $c \geq 0$

		x	y
a^+	1	1	-1
b^+	3	1	1
c	-4	-1	0

- this is a feasible tableau
- it would become unfeasible if we add $c \geq 0$

A feasible tableau denotes a convex polyhedron in $\mathbb{R}^r$ such that $\bar{0} \in S$ ← the polyhedron

- in our case $r=2$ → 2D polygons



Since the sample point does not satisfy $c \geq 0$ $(-x-4 \geq 0)$ we try to move the sample point to increase the value of $-x-4$ (currently is -4)

<u>Simplex idea</u>

- this can be done by translating the sample point along one axis until it hits an enclosing hyperplane
- this step can be repeated a finite number of times

Definition

- Moving the sample point along axis C_j until it reaches the hyperplane corresponding to $R_i \geq 0$ is called pivoting on i, j
- Pivoting is a Gaussian elimination
  - express C_j in terms of R_i and replace in all equations
  - then R_i becomes a column and the sample point is $R_i = 0$
    $C_k = 0 \quad k \neq j$

$$\begin{array}{cc|ccc}
 & & & C(j) & \overset{\text{column } v}{\downarrow} \\
 & & & & R(u) \\
\hline
 & \vdots & & & \\
 & R_i & T_{i0} - \frac{T_{u0} \cdot T_{iv}}{T_{uv}} & T_{ij} - \frac{T_{uj} \cdot T_{iv}}{T_{uv}} & \frac{T_{iv}}{T_{uv}} \\
 & \vdots & & & \\
\text{row } u & C(v) & -\frac{T_{u0}}{T_{uv}} & -\frac{T_{uj}}{T_{uv}} & \frac{1}{T_{uv}} \\
\end{array}$$

__Purpose__ of pivoting

- We choose pivot (u, v) such that
  - the ~~new~~ new sample point is still feasible
  - the sample value of row i grows
    (because we start with $T_{i0} < 0$ and we want to restrict $R_i \geq 0$)
- Note that it only makes sense to choose a pivot that $R_u \geq 0 \rightarrow T_{u0} \geq 0$

__Consider the situation when__

- we try to increase the sample value of R_i
- all entries in Row i are negative and in $+$-restricted columns.
  - since $C(v)$ is $+$-restricted. $\rightarrow \dfrac{T_{u0}}{T_{uv}} \leq 0$
  - since $T_{iv} < 0 \rightarrow \dfrac{T_{u0} \cdot T_{iv}}{T_{uv}} > 0$

    $\rightarrow$ cannot increase sample value of i any further
- We say that row i is __maximized__

Back to the example

		x	y
a^+	1	1	-4
b^+	3	1	1
c	-4	-1	0

- $i=3$ (increase the value of c)
- pivot only in column of x (otherwise $T_{iv}=0$ and no increase)
  - c is correlated only with x
- could choose line of a^+ or b^+

Try pivot b^+, x.

$x = b - y - 3$

$a^+ = 1 + x - 4y = 1 + b - y - 3 - 4y$
$= -2 + b - 5y$

does not have a satisfying sample point.

- translation on x axis to b takes us outside the solution set

ny a^+, x

$x = a + y - 1$

$b = 3 + x + y = 3 + (a + y - 1) + y$

$c = -4 - x = -4 - a - y + 1$

		a^+	y
x	-1	1	1
b^+	2	1	2
c	-3	-1	-1

- Still cannot restrict $c \geq 0$
- Pivot again

- Increase the value of c
    - decrease either a^+ or y
    - but a^+ is at 0 and cannot be decreased (negative entries in $+$-restricted columns are not good)
- Pivot in column of y and row of b^+

$$y = \frac{1}{2} \cdot (b - 2 - a) = \frac{1}{2} \cdot b - 1 - \frac{1}{2} \cdot a$$

$$x = -1 + a + y = -2 + \frac{1}{2} a + \frac{1}{2} b$$

$$c = -3 - a - \frac{1}{2} b + 1 + \frac{1}{2} a = -2 - \frac{1}{2} a - \frac{1}{2} b$$

		a^+	b^+
x	-2	$1/2$	$1/2$
y	-1	$-1/2$	$1/2$
c	-2	$-\frac{1}{2}$	$-1/2$