More On Simplex Review. Generating Equalities
Generating Proofs

Review

- Let S be a set of linear inequalities
- For each inequality $a' + \bar{a} \cdot \bar{x} \geq 0$ we introduce a slack variable $s = a' + \bar{a} \cdot \bar{x}$ and a constraint $s \geq 0$
- We represent equality constraints in a tableau
- Each row i is owned by an expression R_i
  -"- column j -"- C_j

Invariant 1: for each row i

$$\boxed{R_i \simeq T_{i0} + \sum_j T_{ij} \cdot C_j}$$

- We represent inequality constraints by marking owners as +-restricted.

Invariant 2: for each +-restr. owner R_i there is a proof of $R_i \geq 0$. Same for columns.

- Tableau can become unsat when adding a restriction $R_i \geq 0$.

To detect this pivot the tableau to increase T_{i0} to something ≥ 0 before adding the restriction

Invariant 3. for each +-restr. row i $T_{i0} \geq 0$

- Now we know that any tableau satisfying I_3 is satisfiable (set C_j to 0 and R_i to T_{i0})

__Definition__ A row i is maximized at T_{i0} if all non-zero entries T_{ij} are negative and in columns that are +- restricted.

__Theorem__ ~~If a row is not maximized then~~ In a finite number of pivoting steps a row either becomes maximized or T_{i0} is increased past 0 ($T_{i0} \geq 0$)

How to detect equalities

- We say that the columns are linearly independent if

$$S \Rightarrow \bar{a} \cdot \bar{C}_j = 0 \quad \Longrightarrow \quad \bar{a} = \bar{0}$$

- Theorem

If the columns in a feasible tableau are linearly independent then

$$S \Rightarrow R_i = 0 \quad \cancel{\text{only}} \text{ iff } \quad \begin{array}{l} T_{i0} = 0 \\ T_{ij} = 0 \end{array}$$

Proof: "if" is trivial

"only if" $\quad S \Rightarrow R_i = 0$

$$R_i \simeq T_{i0} + \sum T_{ij} \cdot C_j$$

$$\Rightarrow S \Rightarrow T_{i0} + \sum T_{ij} \cdot C_j = 0$$

a) $\bar{0} \in S \Rightarrow T_{i0} = 0$

b) $S \Rightarrow \sum T_{ij} \cdot C_j = 0 \rightarrow T_{ij} = 0$

Corrolary

- if the columns are linearly independent then

1). Never $S \Rightarrow C_j = C_{j'}$ $(j \neq j')$

2) $S \Rightarrow R_i = R_{j'}$ if only if $T_{ij} = T_{ij'}$

3) $S \Rightarrow R_i = C_j$ iff $T_{ij} = 1$

$T_{ik} = 0$ $k \neq j$

(Proof. Consider $S \Rightarrow R_i = C_j$

- add a new row $R_k \simeq R_i - C_j$
- it is going to have the entries

$T_{kj} = T_{ij} - 1$

$T_{kj'} = T_{ij'}$ $j' \neq j$

and $T_{kj} = 0$ $T_{kj'} = 0$ $\square$)

Thus. we can easily detect all equalities between variables if we keep the columns linearly independent.

__Idea__

- start with linearly independent columns
- detect when we introduce dependencies

__Theorem__

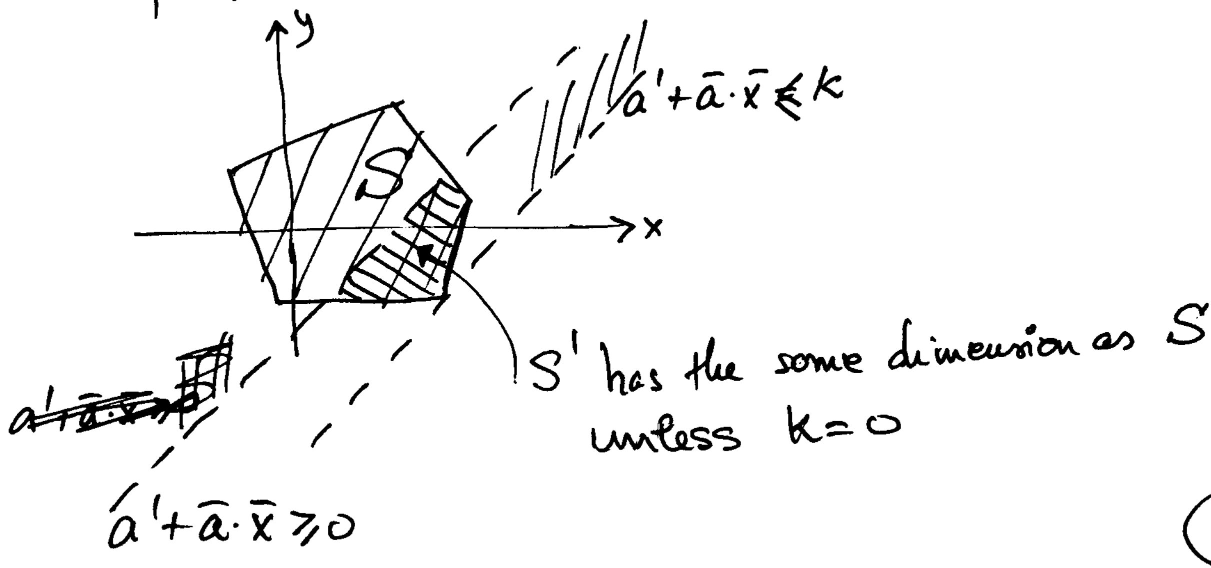
If S has a n-dimensional solution set
(n-linearly independent columns in the tableau)

then $S \wedge a' + \bar{a}\cdot\bar{x} \geq 0$ also has an n-dimensional solution iff

- $a' + \bar{a}\cdot\bar{x}$ is maximized at $k > 0$ in S

- if $a' + \bar{a}\cdot\bar{x}$ is maximized at $k < 0$ in S then $S \wedge a' + \bar{a}\cdot\bar{x} \geq 0$ is unsatisfiable
- if $a' + \bar{a}\cdot\bar{x}$ is maximized at $k = 0$ then $S \wedge \ldots$ has some dependent columns

__Informal proof__ Let $n = 2$



If now i is maximized at 0 and we try to add $R_i \geq 0$ then.

$$S \Rightarrow R_i \leq 0$$

$$S \wedge R_i \geq 0 \Rightarrow R_i = 0$$

Since R_i is maximized $R_i \simeq \sum T_{ij} \cdot C_j$

such that

$T_{ij} \neq 0 \Rightarrow$ $T_{ij} < 0$ and C_j is +-restricted.

- This implies that all such C_j are also $S \Rightarrow C_j = 0$

- The point where $R_i = 0$ meets S is at the intersection of hyperplanes $C_j = 0$ (for C_j such that $T_{ij} < 0$)

- All those columns become fixed to 0 and therefore dependent

→ **We can ignore them in what follows !!**

and thus maintain the tableau in minimal state (with linearly independent columns)

Notation

- Some $+$-restricted rows (columns) are known to be $=0$. We call them $*$-restricted.

- When does a row become $*$-restricted
  - it is $+$-restricted
  - it is maximized at 0.
- But then all C_j such that $T_{ij} \neq 0$ are also $*$-restricted.

Invariant 4

If row i is $*$-restricted then
$T_{i0} = 0$ and $T_{ij} \neq 0 \Rightarrow C_j$ is $*$-restricted

- A C_j becomes $*$-restricted due to a row i becoming $*$-restricted
- $*$-restricted columns are never pivoted (are ignored)

Invariant 5: If column j is $*$-restricted then there exists a $*$-restr. row i such that $T_{ij} < 0$ and $T_{ik} \leq 0$ for all $k > j$

- the $T_{ik} \leq 0$ for all $k > j$ can be achieved by reordering the columns

Note: pivoting such that I_3 is maintained, maintains all the other invariants

The algorithm

simplex(e, prf)

- add row i owning e
- try to increase T_{i0} to ≥ 0 by pivoting
- if row i is maximized at $T_{i0} < 0$

  raise Contra (mkContraProof (i, prf))
- ~~if row~~
- make row i +-restricted with Proof($R_i \geq 0$) = prf
- if row i is maximized at $T_{i0} = 0$
  - make i a $*$-restricted row
  - for all $T_{ij} \neq 0$, make j a $*$-restricted column

/* collect equalities. All non-$*$-restr. columns are linearly independent */

if rows i_1, i_2 differ only in $*$-restr. columns

$R(i_1) = R(i_2)$, mkEqProof ($R(i_1)$, $R(i_2)$)

if row i : $T_{ij} = 1$ $T_{ik} \neq 0 \wedge k \neq j \Rightarrow k$ is $*$-restr.

$R(i) = C(j)$ mkEqProof ($R(i)$, $C(j)$)

Generating proofs

- $e \geq 0$ iff there are $e_i \geq 0 \in S$ (original set of constraints) and factors $a_i \geq 0$ such that $\sum a_i \cdot e_i \simeq \overset{e-k}{\cancel{k}}$ (a constant) and $k \geq 0$.

Theorem Each $*$-restr. column can be written as a negative linear combination of $+$-owners.

Proof: ~~Say k becomes~~ By induction on the sequence of $*$-restrictions.

Say k becomes restricted due to row i becoming $*$-restr. $\rightarrow$ $T_{ik} < 0$ and $R(i)$ is $+$-restr and $R(i)$ is maximized at 0.

$$R_i \simeq \sum_{T_{ij}>0} T_{ij} \cdot \underset{\uparrow\ *\text{-restr.}}{C_j} + \sum_{T_{ij}<0} T_{ij} \cdot \underset{\uparrow\ +\text{-restr.}}{C_j}$$

$$C_k \simeq \underset{<0}{\underbrace{\frac{1}{T_{ik}}}} \cdot \underset{\uparrow\ +\text{-restr}}{R_i} + \sum_{T_{ij}>0} \underset{>0}{\underbrace{\frac{T_{ij}}{-T_{ik}}}} \cdot \underset{\uparrow\ \text{by I.H. a negative lin. comb}}{C_j} + \sum_{T_{ij}<0} \underset{<0}{\underbrace{\frac{T_{ij}}{-T_{ik}}}} \cdot \underset{\uparrow\ +\text{-restr.}}{C_j}$$

Generate a proof that a row maximized at k

$R(i) \leq k$

$$-R_i = -k + \underbrace{\sum_{T_{ij}>0} T_{ij} \cdot \underset{\substack{\uparrow \\ *\text{-restr.}}}{C_j} - \sum_{T_{ij}<0} T_{ij} \cdot \underset{\substack{\uparrow \\ +\text{-restr.}}}{C_j}}_{\text{positive lin. comb. of } +\text{-restr.}}$$

then

$\simeq -R_i + k$

means $-R_i \geq -k$ ∎

Generate a proof that two rows are equal.

$$R_{i_1} - R_{i_2} \simeq \sum_{a_j>0} a_j \cdot C_j + \sum_{a_j<0} a_j \cdot C_j$$

can prove $C_j = 0$ for each $*$-restr. column

$C_j \geq 0$ since $*$-restr. implies $+$-restr

$C_j \leq 0$ because C_j is a negative lin. comb. of $+$-restr.

Undoing Simplex

- Key fact: pivoting does not need to be undone

- Undo
  - delete rows or columns
    - just permute rows so these to be eliminated are last
  - delete * and + restrictions