Nonlinear Optimization for Optimal Control

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[optional] Boyd and Vandenberghe, Convex Optimization, Chapters 9 – 11
[optional] Betts, Practical Methods for Optimal Control Using Nonlinear Programming

Bellman's curse of dimensionality

- n-dimensional state space
- Number of states grows exponentially in n (assuming some fixed number of discretization levels per coordinate)
- In practice
  - Discretization is considered only computationally feasible up to 5 or 6 dimensional state spaces even when using
    - Variable resolution discretization
    - Highly optimized implementations

This Lecture: Nonlinear Optimization for Optimal Control

- Goal: find a sequence of control inputs (and corresponding sequence of states) that solves:

$$\begin{aligned} \min_{u,x} \quad & \sum_{t=0}^{H} g(x_t, u_t) \\ \text{subject to} \quad & x_{t+1} = f(x_t, u_t) \quad \forall t \\ & u_t \in \mathcal{U}_t \quad \forall t \\ & x_t \in \mathcal{X}_t \quad \forall t \end{aligned}$$

- Generally hard to do. We will cover methods that allow to find a local minimum of this optimization problem.
- Note: iteratively applying LQR is one way to solve this problem if there were no constraints on the control inputs and state

Outline

- **Unconstrained minimization**
  - **Gradient Descent**
  - Newton's Method
- Equality constrained minimization
- Inequality and equality constrained minimization

Unconstrained Minimization

$$\boxed{\min_x f(x) \qquad (1)}$$

(Implicitly assumed x can be chosen from the entire domain of f, often $\mathbb{R}^n$.)

- If x* satisfies:

$$\begin{aligned} \nabla_x f(x^*) &= 0 \quad (2) \\ \nabla_x^2 f(x^*) &\succeq 0 \quad (3) \end{aligned}$$

then x* is a local minimum of f.

- In simple cases we can directly solve the system of n equations given by (2) to find candidate local minima, and then verify (3) for these candidates.
- In general however, solving (2) is a difficult problem. Going forward we will consider this more general setting and cover numerical solution methods for (1).

Steepest Descent

- Idea:
  - Start somewhere
  - Repeat: Take a small step in the steepest descent direction

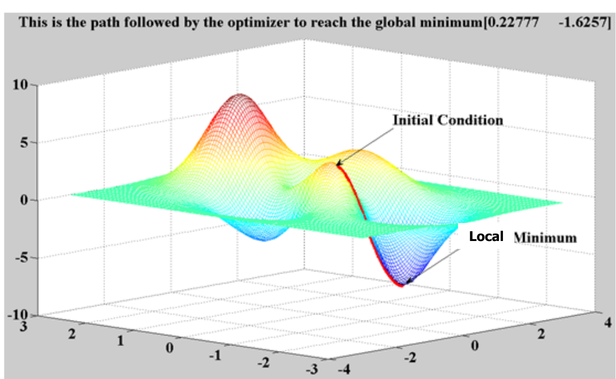


Figure source: Mathworks

Steep Descent

- Another example, visualized with contours:

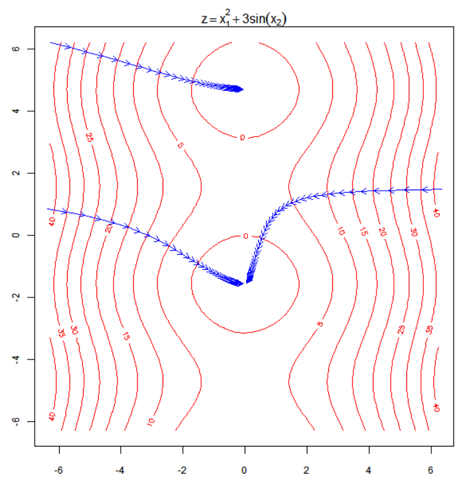


Figure source: yihui.name

Steepest Descent Algorithm

1. Initialize x
2. Repeat
    1. Determine the steepest descent direction Δx
    2. Line search. Choose a step size t > 0.
    3. Update. x := x + t Δx.
3. Until stopping criterion is satisfied

What is the Steepest Descent Direction?

Assuming a smooth function, we have that

$$f(x_0 + \Delta x) \approx f(x_0) + \nabla_x f(x_0)^\top \Delta x$$

The (locally at x_0) direction of steepest descent is given by:

$$\begin{aligned} \Delta x^* &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} f(x_0) + \nabla_x f(x_0)^\top \Delta x \\ &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} \nabla_x f(x_0)^\top \Delta x \end{aligned}$$

As we have all $a, b \in \mathbb{R}^n$ that $\min_{b : \|b\|_2 = 1} a^\top b$ is achieved for $b = -\frac{a}{\|a\|_2}$, we have that the steepest descent direction

$$\Delta x^* = -\nabla_x f(x_0)$$

Stepsize Selection: Exact Line Search

$$t = \arg \min_{s \geq 0} f(x + s\Delta x)$$

- Used when the cost of solving the minimization problem with one variable is low compared to the cost of computing the search direction itself.

Stepsize Selection: Backtracking Line Search

- Inexact: step length is chose to approximately minimize f along the ray {x + t Δx | t $\geq$ 0}

Backtracking Line Search.
given a descent direction Δx for f at $x \in \mathbf{dom} f$, $\alpha \in (0, 0.5), \beta \in (0, 1)$.
$t := 1$
while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^\top \Delta x, t := \beta t$.

Stepsize Selection: Backtracking Line Search

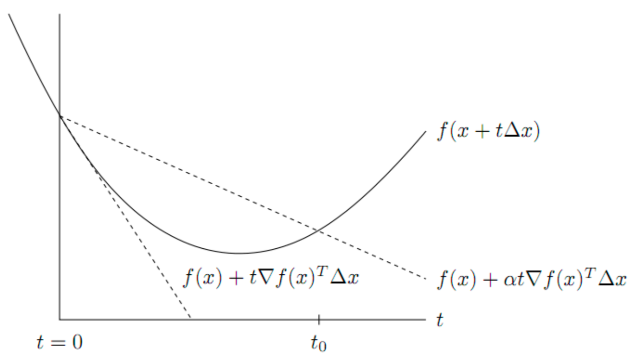


Figure 9.1 *Backtracking line search.* The curve shows f, restricted to the line over which we search. The lower dashed line shows the linear extrapolation of f, and the upper dashed line has a slope a factor of α smaller. The backtracking condition is that f lies below the upper dashed line, *i.e.*, $0 \leq t \leq t_0$.

Figure source: Boyd and Vandenberghe

Gradient Descent Method

Algorithm 9.3 *Gradient descent method.*

given a starting point $x \in \mathbf{dom}\, f$.

repeat

1. $\Delta x := -\nabla f(x)$.
2. *Line search.* Choose step size t via exact or backtracking line search.
3. *Update.* $x := x + t\Delta x$.

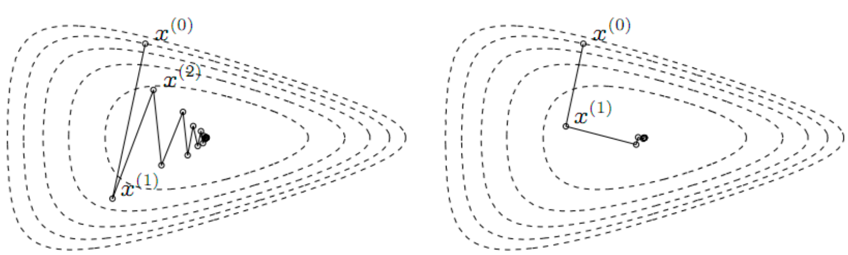
until stopping criterion is satisfied.

The stopping criterion is usually of the form $\|\nabla f(x)\|_2 \leq \eta$, where η is small and positive. In most implementations, this condition is checked after step 1, rather than after the update.

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



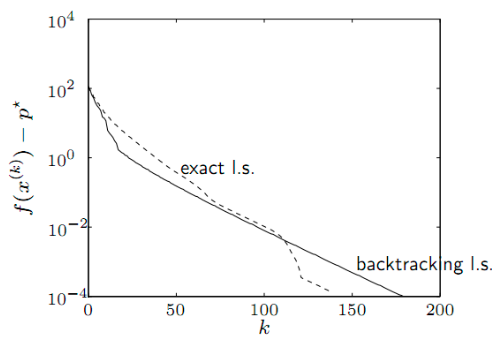
backtracking line search   exact line search

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 2

a problem in $\mathbf{R}^{100}$

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



'linear' convergence, *i.e.*, a straight line on a semilog plot

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

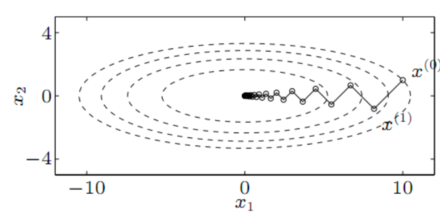
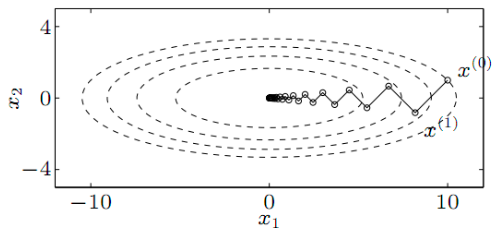
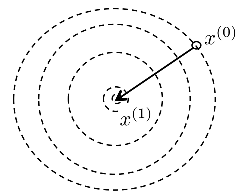


Figure source: Boyd and Vandenberghe

Gradient Descent Convergence



Condition number = 10



Condition number = 1

- For quadratic function, convergence speed depends on ratio of highest second derivative over lowest second derivative ("condition number")
- In high dimensions, almost guaranteed to have a high (=bad) condition number
- Rescaling coordinates (as could happen by simply expressing quantities in different measurement units) results in a different condition number

Outline

- **Unconstrained minimization**
  - Gradient Descent
  - **Newton's Method**
- Equality constrained minimization
- Inequality and equality constrained minimization

Newton's Method

- 2^{nd} order Taylor Approximation rather than 1^{st} order:

$$f(x+\Delta x) \approx f(x) + \nabla f(x)^\top \Delta x + \frac{1}{2}\Delta x^\top \nabla^2 f(x) \Delta x$$

assuming $\nabla^2 f(x) \succeq 0$, the minimum of the 2^{nd} order approximation is achieved at: $\Delta x_{\mathrm{nt}} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$

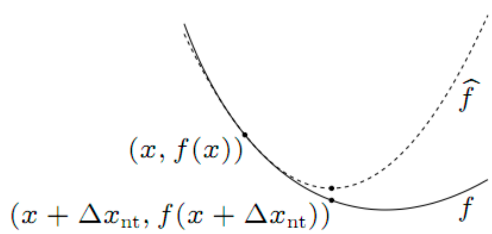


Figure source: Boyd and Vandenberghe

Newton's Method

Algorithm 9.5 *Newton's method.*

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.

repeat

1. *Compute the Newton step and decrement.*
   $\Delta x_{\mathrm{nt}} := -\nabla^2 f(x)^{-1} \nabla f(x)$; $\quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

Figure source: Boyd and Vandenberghe

Affine Invariance

- Consider the coordinate transformation y = A x
- If running Newton's method starting from $x^{(0)}$ on f(x) results in

  $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$
- Then running Newton's method starting from $y^{(0)} = A\, x^{(0)}$ on $g(y) = f(A^{-1} y)$, will result in the sequence

  $y^{(0)} = A\, x^{(0)}, y^{(1)} = A\, x^{(1)}, y^{(2)} = A\, x^{(2)}, \ldots$
- Exercise: try to prove this.

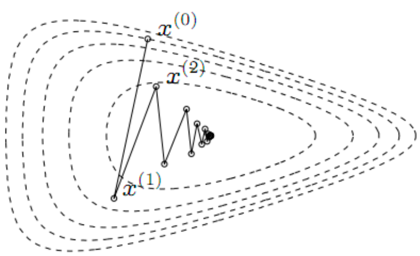
Newton's method when we don't have $\nabla^2 f(x) \succeq 0$

- Issue: now Δx_{nt} does not lead to the local minimum of the quadratic approximation --- it simply leads to the point where the gradient of the quadratic approximation is zero, this could be a maximum or a saddle point
- Three possible fixes, let $X \Lambda X^\top = \nabla^2 f(x)$ be the eigenvalue decomposition.
  - Fix 1: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \max(0, \Lambda_{i,i})$.
  - Fix 2: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \Lambda_{i,i} + (-1) * \min_j \Lambda_{j,j}$
  - Fix 3: Use a gradient descent step, rather than a Newton step, in the current iteration.

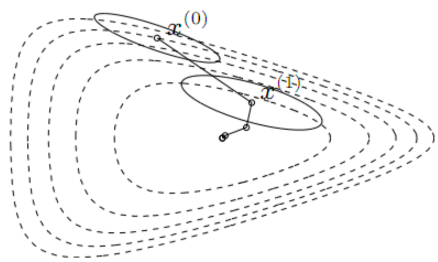
In my experience Fix 2 works best.

Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



gradient descent with
backtracking line search



Newton's method with
backtracking line search

Figure source: Boyd and Vandenberghe

Example 2

a problem in $\mathbf{R}^{100}$

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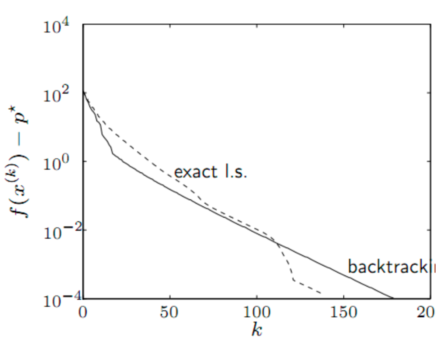
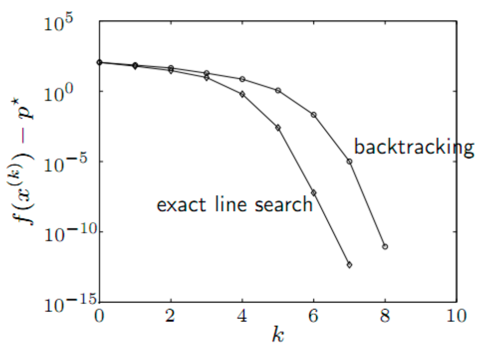


gradient descent



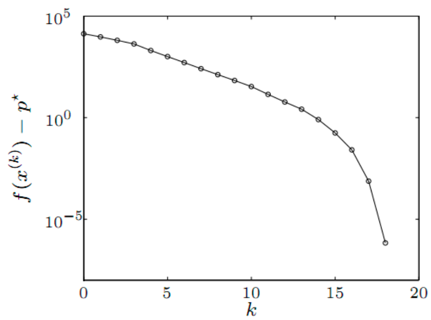
Newton's method

Figure source: Boyd and Vandenberghe

Larger Version of Example 2

example in $\mathbf{R}^{10000}$ (with sparse a_i)

$$f(x) = -\sum_{i=1}^{10000} \log(1 - x_i^2) - \sum_{i=1}^{100000} \log(b_i - a_i^T x)$$



- backtracking parameters $\alpha = 0.01$, $\beta = 0.5$.
- performance similar as for small examples

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

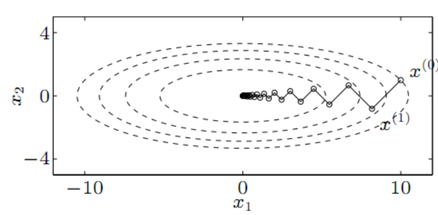
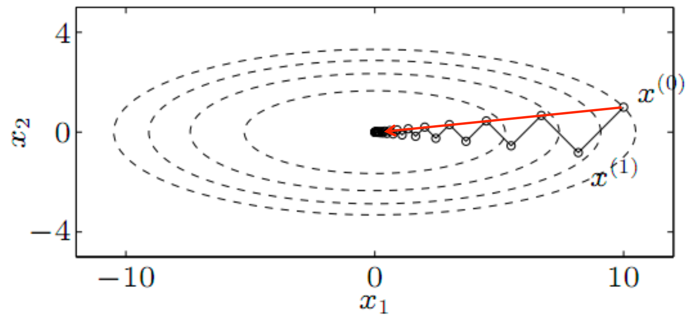


Figure source: Boyd and Vandenberghe

Example 3



- Gradient descent
- Newton's method (converges in one step if f convex quadratic)

Quasi-Newton Methods

- Quasi-Newton methods use an approximation of the Hessian
  - Example 1: Only compute diagonal entries of Hessian, set others equal to zero. Note this also simplfies computations done with the Hessian.
  - Example 2: natural gradient --- see next slide

Natural Gradient

- Consider a standard maximum likelihood problem:

$$\max_\theta f(\theta) = \max_\theta \sum_i \log p(x^{(i)}; \theta)$$

- Gradient:

$$\frac{\partial f(\theta)}{\partial \theta_p} = \sum_i \frac{\partial \log p(x^{(i)}; \theta)}{\partial \theta_p} = \sum_i \frac{\partial p(x^{(i)}; \theta)}{\partial \theta_p} \frac{1}{p(x^{(i)}; \theta)}$$

- Hessian:

$$\frac{\partial^2 f(\theta)}{\partial \theta_q \partial \theta_p} = \sum_i \frac{\partial^2 p(x^{(i)}; \theta)}{\partial \theta_q \partial \theta_p} \frac{1}{p(x^{(i)}; \theta)} - \frac{\partial p(x^{(i)}; \theta)}{\partial \theta_q} \frac{1}{p(x^{(i)}; \theta)} \frac{\partial p(x^{(i)}; \theta)}{\partial \theta_p} \frac{1}{p(x^{(i)}; \theta)}$$

$$\nabla^2 \log f(\theta) = \sum_i \frac{\nabla^2 p(x^{(i)}; \theta)}{p(x^{(i)}; \theta)} - \left(\nabla \log p(x^{(i)}; \theta)\right) \left(\nabla \log p(x^{(i)}; \theta)\right)^\top$$

- Natural gradient only keeps the 2nd term
  1: faster to compute (only gradients needed); 2: guaranteed to be negative definite; 3: found to be superior in some experiments

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