Lecture 10

1 Cheeger's inequality

In the last lecture we introduced the notion of edge expansion, eigenvalues of the adjacency matrix and the averaging interpretation of the action of the normalized adjacency matrix M and stated Cheeger's inequality that relates the spectral gap to the expansion.

$$\frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} \tag{1}$$

Today we will prove the left side of Cheeger's inequality, the proof of the right side of the inequality is harder and we will see it in a future lecture.

Why is the left side of Cheeger easier? The left side of Cheeger's inequality is equivalent to proving that $\lambda_2 \geq 1 - 2h(G)$. It is easy to prove an inequality of the form $\lambda_2 \geq c$ using the Rayleigh quotient characterization of the second eigenvalue from the previous lecture,

$$\lambda_2 = \max_{x \perp 1} \frac{x^T M x}{x^T x} \tag{2}$$

In order to prove that $\lambda_2 \geq c$ it suffices to find a vector $v \in \mathbb{R}^n, v \perp \vec{1}$ such that the Rayleigh quotient $\frac{x^T M x}{x^T x} \geq c$. The averaging interpretation of the action of M is useful for bounding the Rayleigh quotient.

Proof idea: Given a partition $(S, \overline{S})$ of the vertices of G with edge expansion $h(S)$ the proof idea is to find vector $v \in \mathbb{R}^n, v \perp \vec{1}$ with Rayleigh quotient at least $1-2h(S)$. Applying the argument to the sparsest cut in G yields the left side of Cheeger's inequality,

$$\lambda_2 \geq 1 - 2h(G) \tag{3}$$

Claim 1
Given a partition $(S, \overline{S})$ of the vertices of G with $|S| \leq n/2$, define vector v such that $v_i = -|\overline{S}|$ for $i \in S$ and $v_i = |S|$ for $i \in \overline{S}$.

$$\frac{v^T M v}{v^T v} \geq 1 - 2h(S)$$

Proof: The vector $v \perp \vec{1}$ by design as the vertices in S contribute $-|S||\overline{S}|$ to $\sum v_i$ which is cancelled by the $|S||\overline{S}|$ contributed by vertices in $\overline{S}$. In order to bound the Rayleigh quotient, we compute the quantities $v^T v$ and $v^T M v$,

$$v^T v = \sum_{i \in S} |\overline{S}|^2 + \sum_{i \in \overline{S}} |S|^2 = |S|.|\overline{S}|.(|S| + |\overline{S}|) = n|S|.|\overline{S}| \tag{4}$$

If there are no edges in G crossing the partition $(S, \overline{S})$ then $Mv = v$ and $v^T Mv = v^T v$ by the averaging interpretation of the action of M. Consider the effect of adding an edge (i, j) across the partition. One of the terms in the average $\frac{1}{d}\sum_{k\sim i} v_k$ changes from $-|\overline{S}|$ to $|S|$, this results in a net increase of $\frac{|S|+|\overline{S}|}{d} = \frac{n}{d}$ in the average value. Arguing similarly, we find that the average value $\frac{1}{d}\sum_{k\sim j} v_k$ decreases by $\frac{n}{d}$.

Adding an edge (i, j) across the partition changes $Mv_i \to Mv_i + \frac{n}{d}$ and $Mv_j \to Mv_j - \frac{n}{d}$. The inner product between v and Mv changes by $-(|\overline{S}| + |S|)\frac{n}{d} = -\frac{n^2}{d}$ for the addition of every edge across $(S, \overline{S})$. Therefore,

$$\begin{aligned} v^T Mv &= v^T v - \frac{n^2}{d}|E(S, \overline{S})| \\ &= n|S||\overline{S}| - n^2|S|h(S) \end{aligned} \tag{5}$$

The equality $|E(S, \overline{S})| = d|S|h(S)$ follows from the definition of edge expansion. The value of the Rayleigh quotient is,

$$\frac{v^T Mv}{v^T v} = \frac{n|S||\overline{S}| - n^2|S|h(S)}{n|S||\overline{S}|} = 1 - \frac{n}{|\overline{S}|}h(S) \geq 1 - 2h(S) \tag{6}$$

□

1.1 The spectral gap as a relaxation of conductance

Another perspective on Cheeger's inequality is the observation that the spectral gap $(1 - \lambda_2)$ is a relaxation of the optimization problem of computing the conductance $\phi(G)$. The spectral gap can be written in terms of the Rayleigh quotient,

$$\begin{aligned} 1 - \lambda_2 &= \min_{x\perp 1} \frac{x^T x - x^T Mx}{x^T x} = \min_{x\perp 1} \frac{d\sum_i x_i^2 - \sum_{ij} 2A_{ij}x_i x_j}{d\sum_i x_i^2} \\ &= \min_{x\perp 1} \frac{\sum_{ij} A_{ij}(x_i - x_j)^2}{d\sum_i x_i^2} \end{aligned} \tag{7}$$

The sum of the entries of x is equal to 0 as $x \perp 1$ so we have $(\sum x_i)^2 = \sum x_i^2 + \sum 2x_i x_j = 0$. The expression in the denominator of the above expression can be rearranged to obtain $d\sum_i x_i^2 = \frac{d}{n}\sum_{i,j}(x_i - x_j)^2$,

$$1 - \lambda_2 = \min_{x\perp 1} \frac{n\sum_{ij} A_{ij}(x_i - x_j)^2}{d\sum_{ij}(x_i - x_j)^2} \tag{8}$$

The expression for the spectral gap is invariant under shifting all the coordinates of x by a constant, so the constraint $x \perp 1$ can be changed to $x \in \mathbb{R}^n \setminus 0$. If x is restricted to the characteristic vector of a cut $\{0, 1\}^n \setminus 0$ the value of the expression (8) is the conductance of the cut defined by x. The conductance $\phi(G)$ can therefore be viewed as a relaxation of the spectral gap,

$$\phi(G) = \min_{S\subset[n]} \frac{nE(S, \overline{S})}{d|S||\overline{S}|} = \min_{x\in\{0,1\}^n\setminus 0} \frac{n\sum_{ij} A_{ij}(x_i - x_j)^2}{d\sum_{ij}(x_i - x_j)^2} \tag{9}$$

The conductance is obtained by minimizing the expression (8) over characteristic vectors of cuts in $\{0,1\}^n \setminus 0$ while the spectral gap is obtained by minimizing the same expression over $\mathbb{R}^n \setminus 0$. It follows that $\phi(G) \geq 1 - \lambda_2$ and using $2h(G) \geq \phi(G)$ we have another proof of the left side of Cheeger's inequality.