Adaptive Isogeometric Analysis by Local h-Refinement with T-splines

Michael Dörfel[1], Bert Jüttler[2], Bernd Simeon[1]
[1]TU Munich, Germany [2]JKU Linz, Austria

SIMAI, Minisymposium M13

Outline

- Preliminaries: Galerkin projection, Isogeometric approach
- Tensor-product splines and T-splines
- Examples
- Future Work: EXCITING

Outline

- **Preliminaries: Galerkin projection, Isogeometric approach**
- Tensor-product splines and T-splines
- Examples
- Future Work: EXCITING

The problem

L is a second order elliptic operator

Ω is a Lipschitz domain with boundary $\Gamma = \Gamma_D \dot{\cup} \Gamma_N$

Solve

$$Lu = f \quad \textbf{in } \Omega$$

for u, subject to Dirichlet and von Neumann boundary conditions

$$u = 0 \textbf{ on } \Gamma_D, \quad \langle \nabla u, \mathbf{n} \rangle = h \textbf{ on } \Gamma_N,$$

where f and h are given data.

The weak form of the problem

Find $u \in V$ **such that**

$$a(u, v) = l(v) \quad \textbf{for all } v \in V$$

where

$$V = \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}$$

$a : V \times V \to \mathbb{R}$ is the symmetric bilinear form corresponding to L.

The linear functional $l : V \to \mathbb{R}$ contains the right-hand side f and the Neumann function h.

The Galerkin projection...

...replaces V by the n-dimensional space

$$S_h = \text{span}\{\phi_1, \ldots, \phi_n\} \subset V.$$

Find $u_h \in S_h$ such that

$$a(u_h, v) = l(v) \quad \textbf{for all } v \in S_h$$

$\Leftrightarrow$ **Solve $Aq = b$ where**

$$A = (a(\phi_i, \phi_j))_{i,j=1,\ldots,n} \quad \textbf{is the stiffness matrix}$$

$$b = (l(\phi_1), \ldots, l(\phi_m)) \quad \textbf{is the right-hand side}$$

and the vector q contains the coefficients of u_h,

$$u_h = \sum_{i=0}^{n} q_i \phi_i$$

Desirable properties

1. **Convergence:** Refinement ($h \to 0$) implies convergence to the exact solution. While local refinement is preferred in practice, uniform refinement is the basis for standard convergence proofs.

2. **Regularity:** We aim at conforming methods with basis functions at least in $H^1(\Omega)$. In contrast to conventional FEM wisdom, additional global smoothness is regarded as beneficial.

3. **Support:** The basis functions should have a small and compact support.

4. **Accurate representation of geometry:** complex geometries should be exactly resolved already on coarse grids.

The isoparametric approach

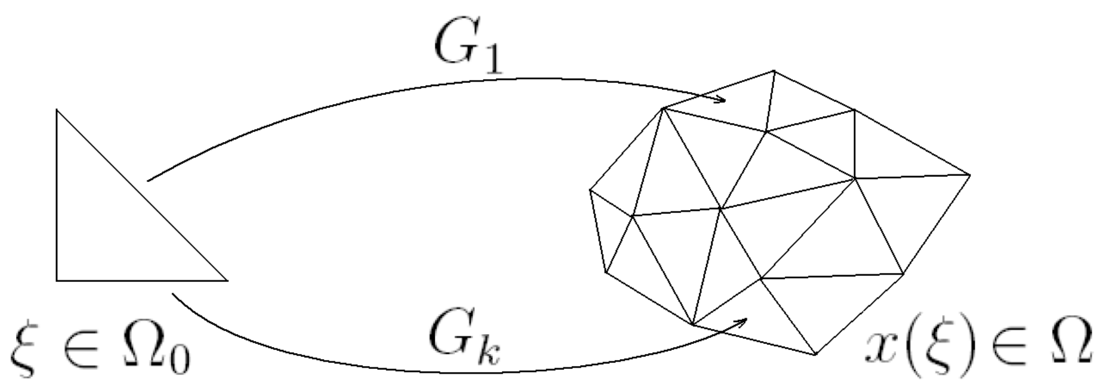
Ω_0 **standard geometry** (e.g. a triangle)

$N_1, \ldots, N_m$ **shape functions** (e.g. [linear] polynomials)

The domain Ω is partitioned into a mesh T_k with **grid points** x_i^k (e.g. corners of the triangle). Its elements (triangles) are described by **geometry functions**

$$G_k(\xi) = \sum N_j(\xi) x_j^k$$

The **basis functions** of $S \subset V$ are $\Phi_i = N_j \circ G_k^{-1}$



The isoparametric approach (continued)

h **refinement:** split the elements

p **refinement:** increase the degree of the shape functions N_i

Well-established **a posteriori error estimators** are available to guide the refinement.

The global smoothness of the functions ϕ_i is C^0

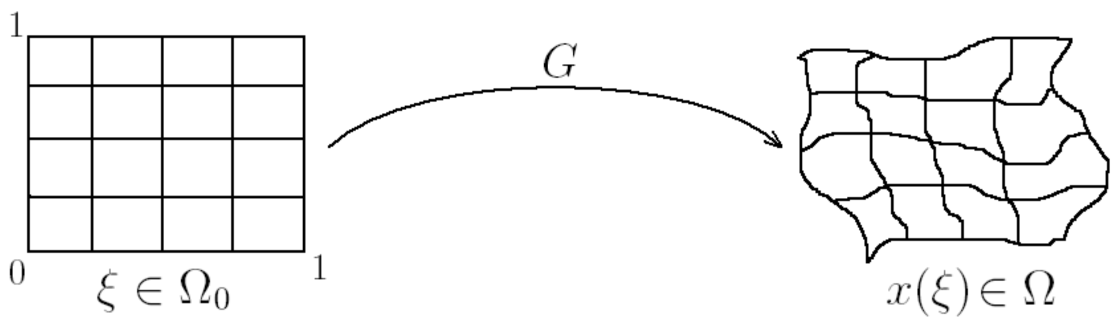
Most obvious drawback: **no exact geometry description**

Isogeometric Analysis

(T. Hughes et al. 2005) uses **only one global geometry function**

$$G : \Omega_0 = [0,1]^2 \to \Omega$$

$$G(\xi) = \sum_i N_i(\xi) P_i \text{ with tensor-product B-splines } N_i$$



The basis functions of $S \subset V$ are $\phi_i = N_i \circ G^{-1}$

h **refinement:** knot insertion

p **refinement:** degree elevation

k **refinement:** do both

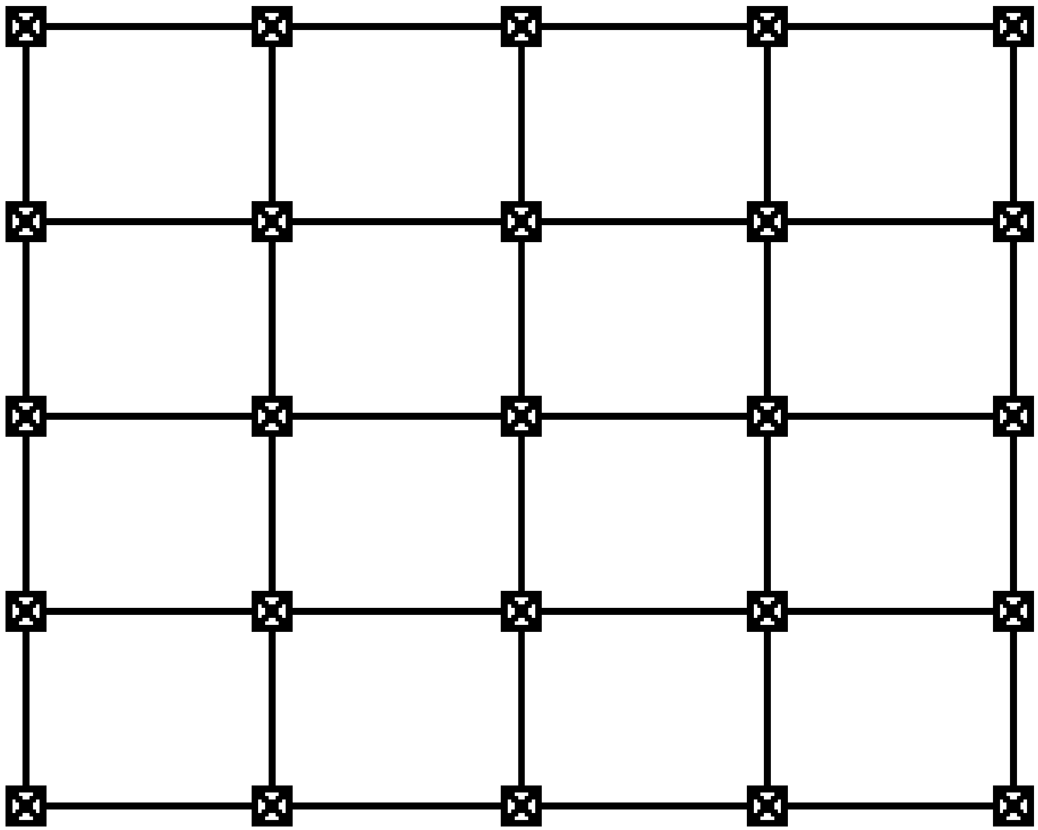
Drawback of tensor-product splines: All refinements act globally!

Outline

- Preliminaries: Galerkin projection, Isogeometric approach
- **Tensor-product splines and T-splines**
- Examples
- Future Work: EXCITING

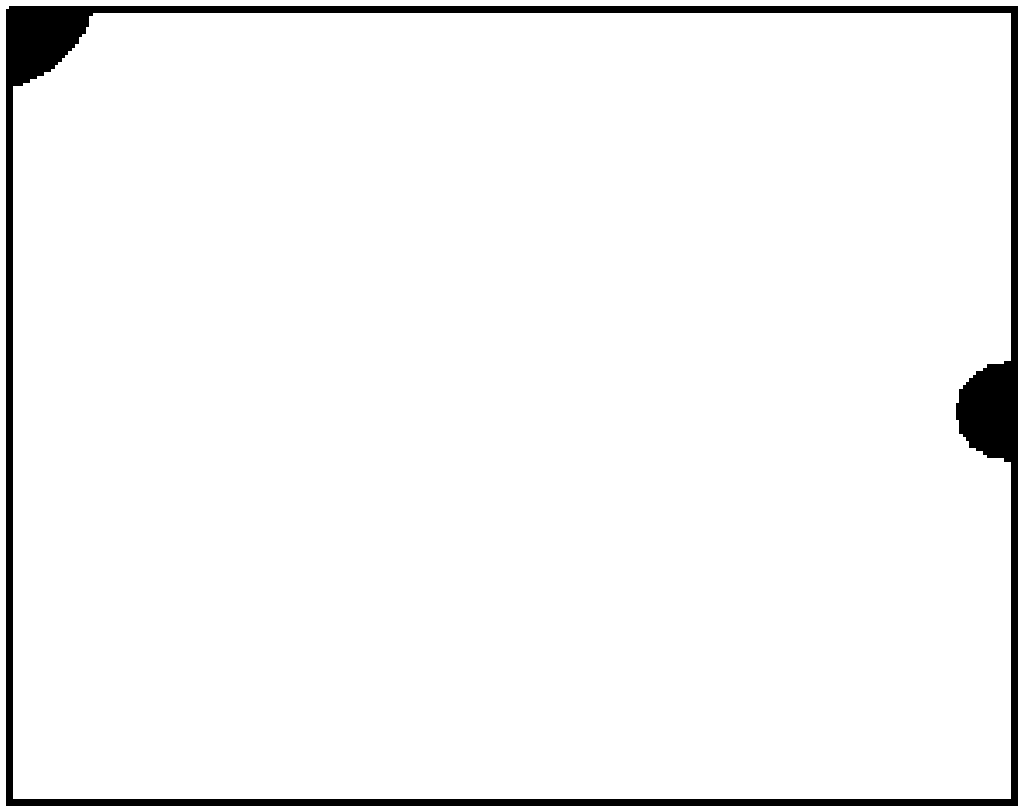
From tensor-product B-splines to T-splines

Coarse tensor-product B-spline



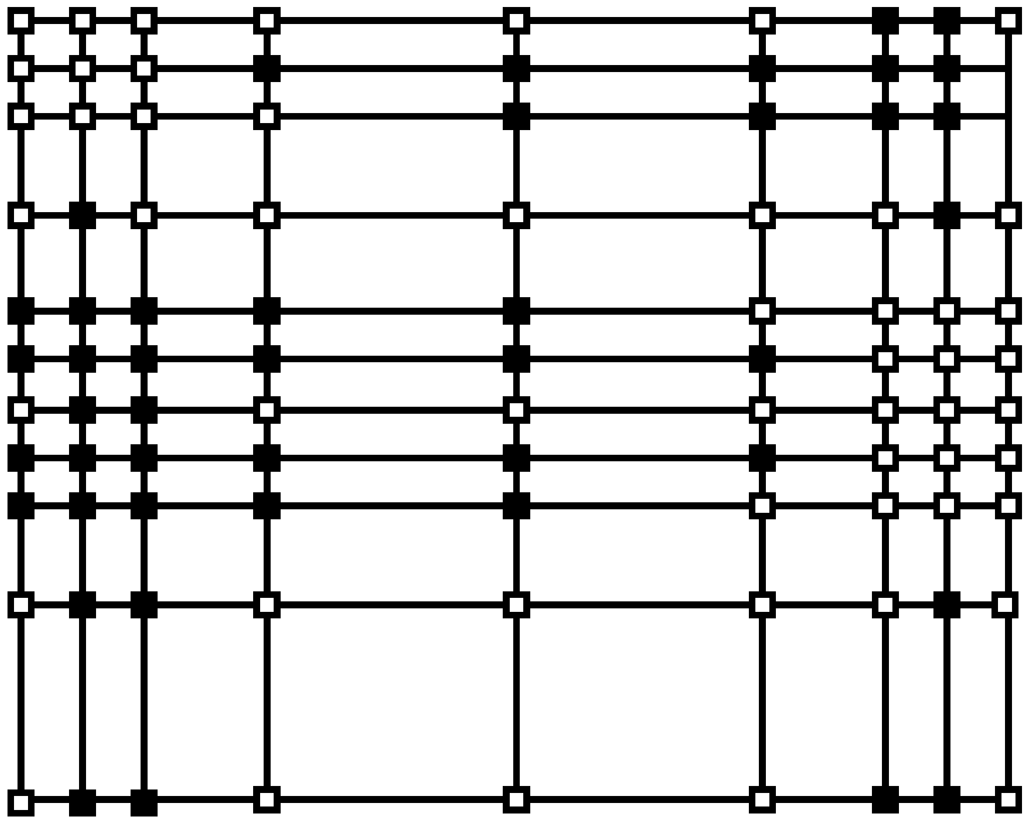
From tensor-product B-splines to T-splines

Error estimator indicates necessary refinements



From tensor-product B-splines to T-splines

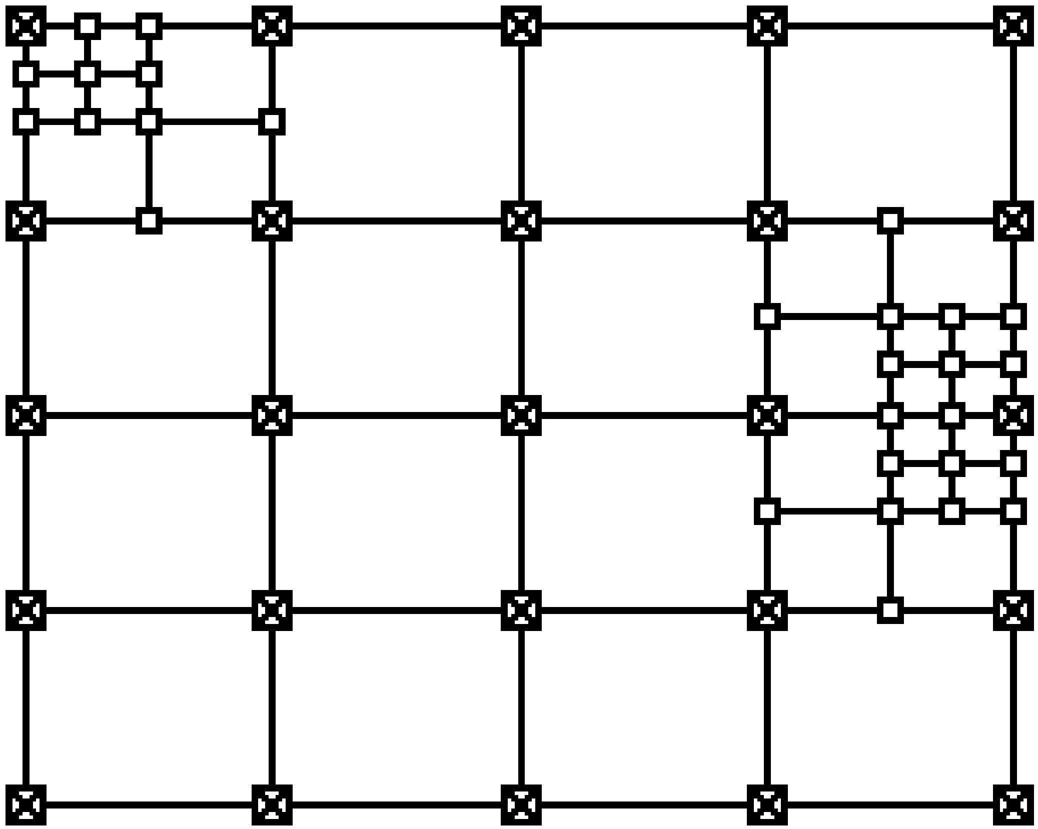
Tensor-product splines require global refinements



black dots: control points associated with “unwanted” basis functions

From tensor-product B-splines to T-splines

T-splines support local refinement



Locally refinable tensor-product splines

Forsey & Bartels 1995: hierarchical splines

Weller & Hagen 1995: splines with knot line segments

Greiner & Hormann 1997: scattered data fitting with hierarchical splines

Sederberg et al. 2003, 2004: T-splines

Deng, Cheng and Feng 2006: Dimensions of certain splines (e.g. C^1, degree (3,3)) over T-meshes

Locally refinable tensor-product splines

Forsey & Bartels 1995: hierarchical splines

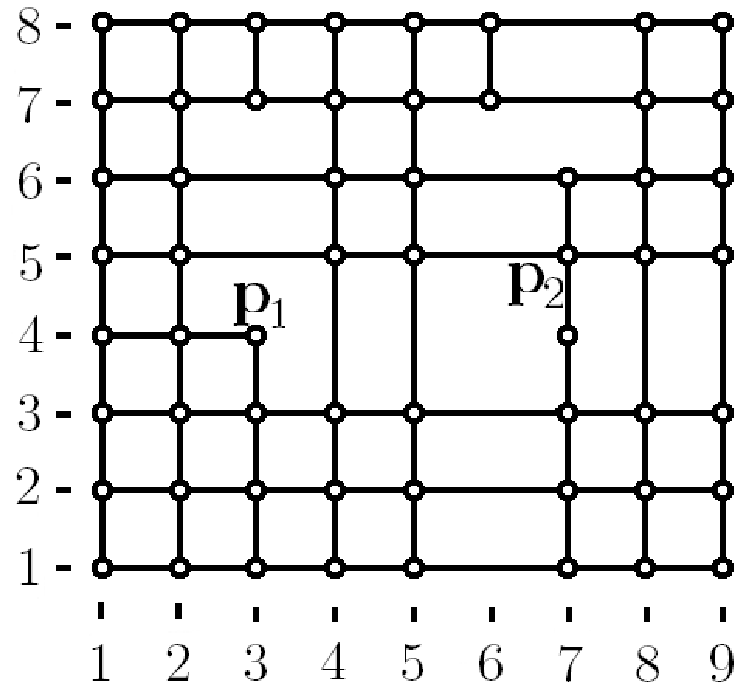
Weller & Hagen 1995: splines with knot line segments

Greiner & Hormann 1997: scattered data fitting with hierarchical splines

Sederberg et al. 2003, 2004: T-splines

Deng, Cheng and Feng 2006: Dimensions of certain splines (e.g. C^1, degree (3,3)) over T-meshes

T-meshes



The edges intersect in grid points.

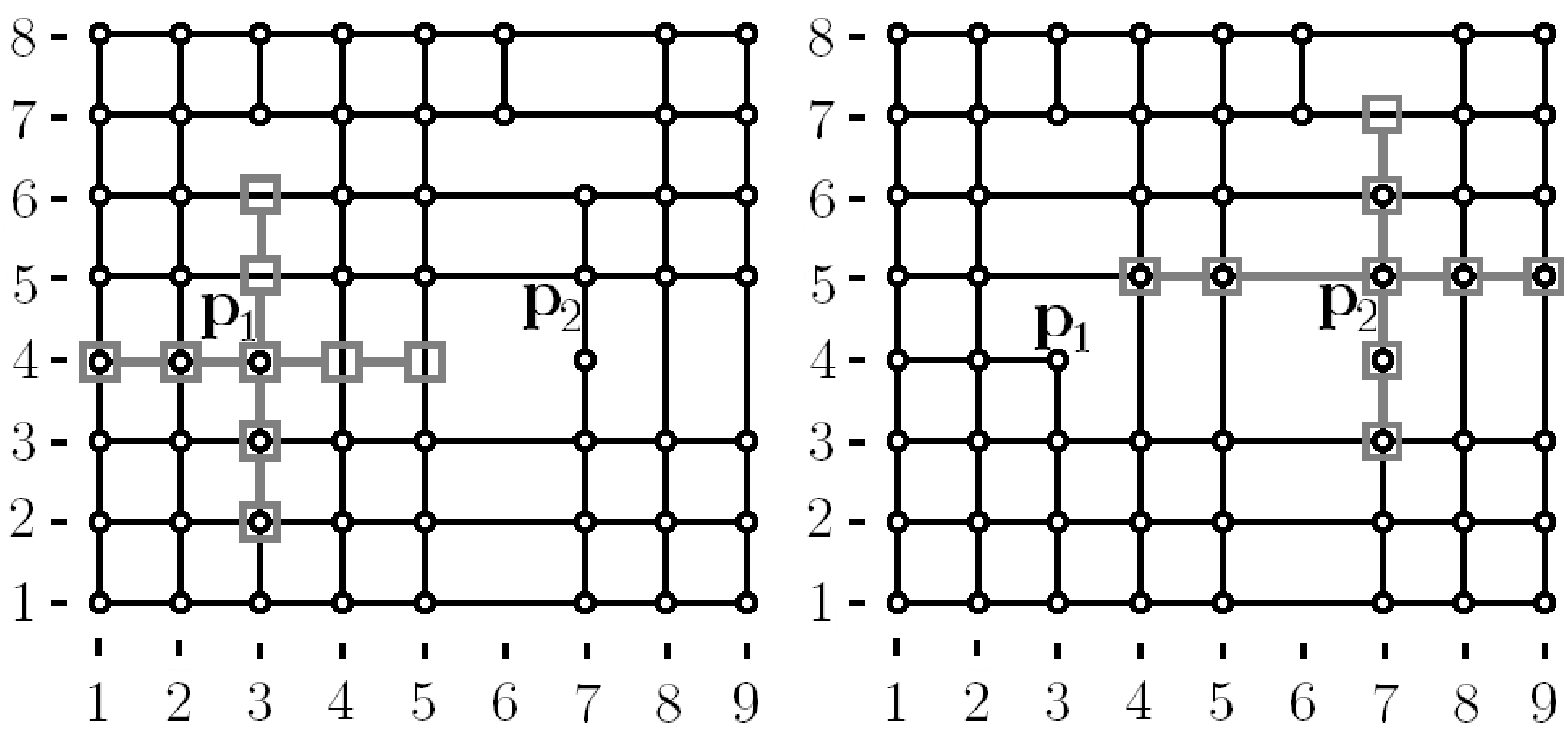
No additional edges connecting grid points can be added.

The T-mesh partitions the box Ω_I into regular polygons (patches).

Blending functions associated with T-meshes...

...are products of B-splines

$$N_{i,j}(s,t) = B_{\sigma(i)}(s)B_{\tau(j)}(t)$$



Refinement

The patches in the T-mesh marked by the **error estimator** are split.

We use a **state-of-the-art error estimator**, which is based on hierarchical bases and bubble functions.

Outline

- Preliminaries: Galerkin projection, Isogeometric approach

- Tensor-product splines and T-splines

- **Examples**

- Future Work: EXCITING

Examples

The three examples are adopted from T. Hughes et al. (2005).

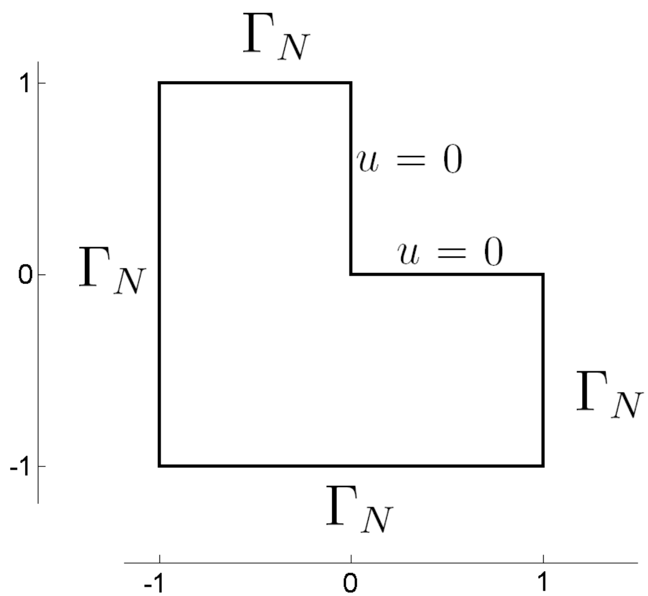
1: Stationary heat conduction

2: Linear Elasticity

3: Advection Dominated Advection–Diffusion

Example 1: Stationary heat conduction

Solve the Laplace equation $\Delta u = 0$ on



subject to

$$\langle \nabla u, \mathbf{n} \rangle = \langle \nabla f, \mathbf{n} \rangle \text{ on } \Gamma_N \quad u = 0 \text{ on } \Gamma_D$$

where f is the exact solution

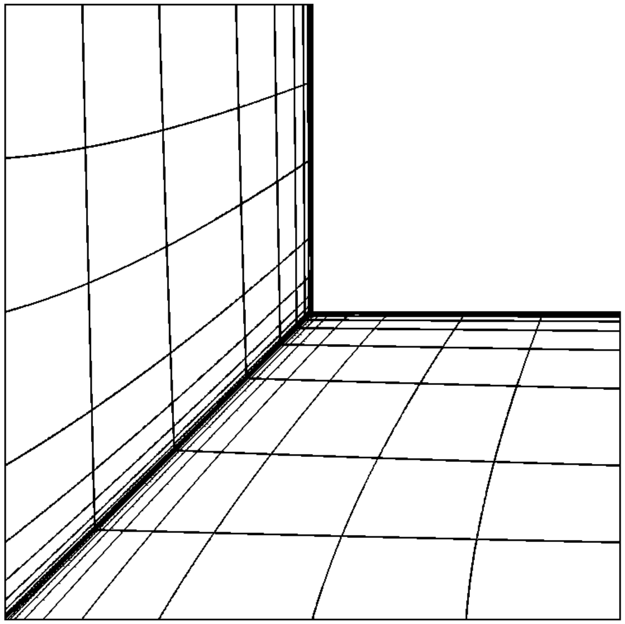
$$f(r, \phi) = r^{2/3} \sin\left(\frac{2\phi - \pi}{3}\right)$$

Example 1: Stationary heat conduction

Description of the domain: 2 biquadratic patches, joined C^0 along the diagonal (no singular parameterization, as in Hughes et al, 2005)

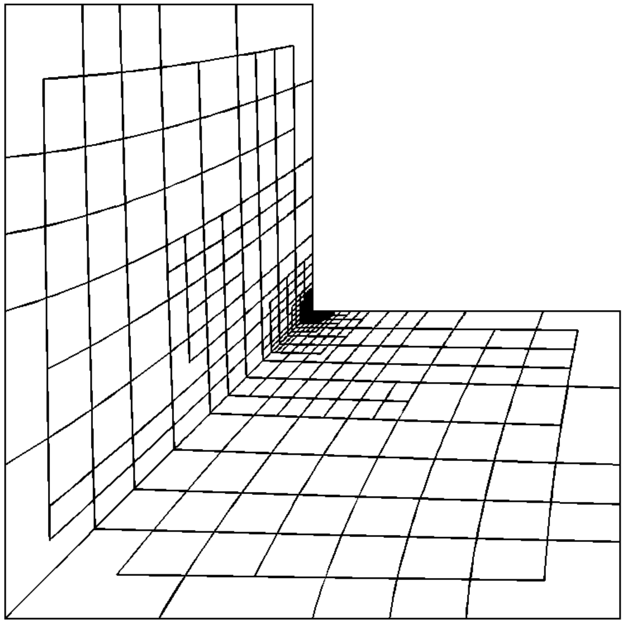
We compared **uniform** refinement with

"Rule of thumb" refinement



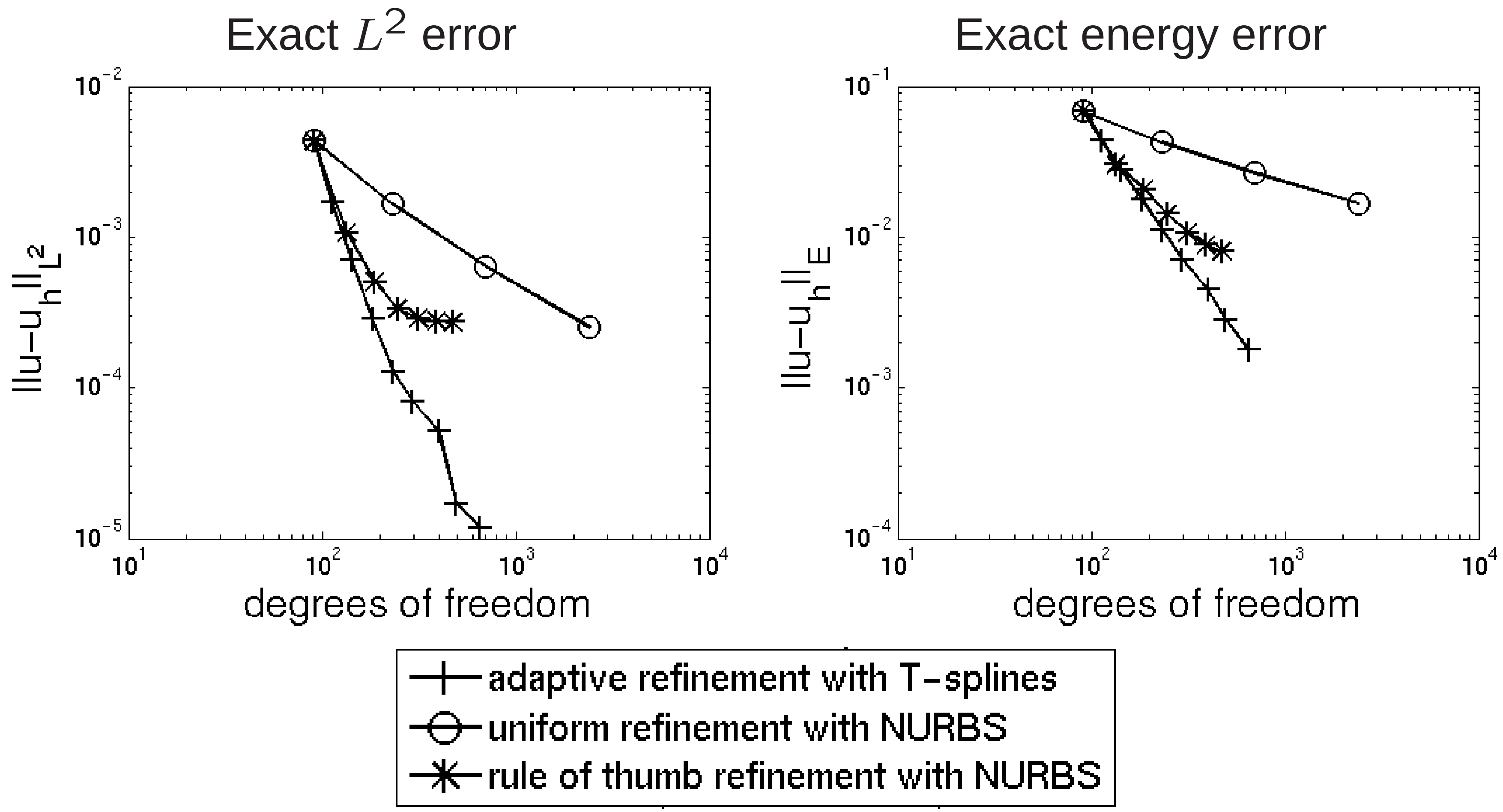
475 dof

adaptive Refinement



652 dof

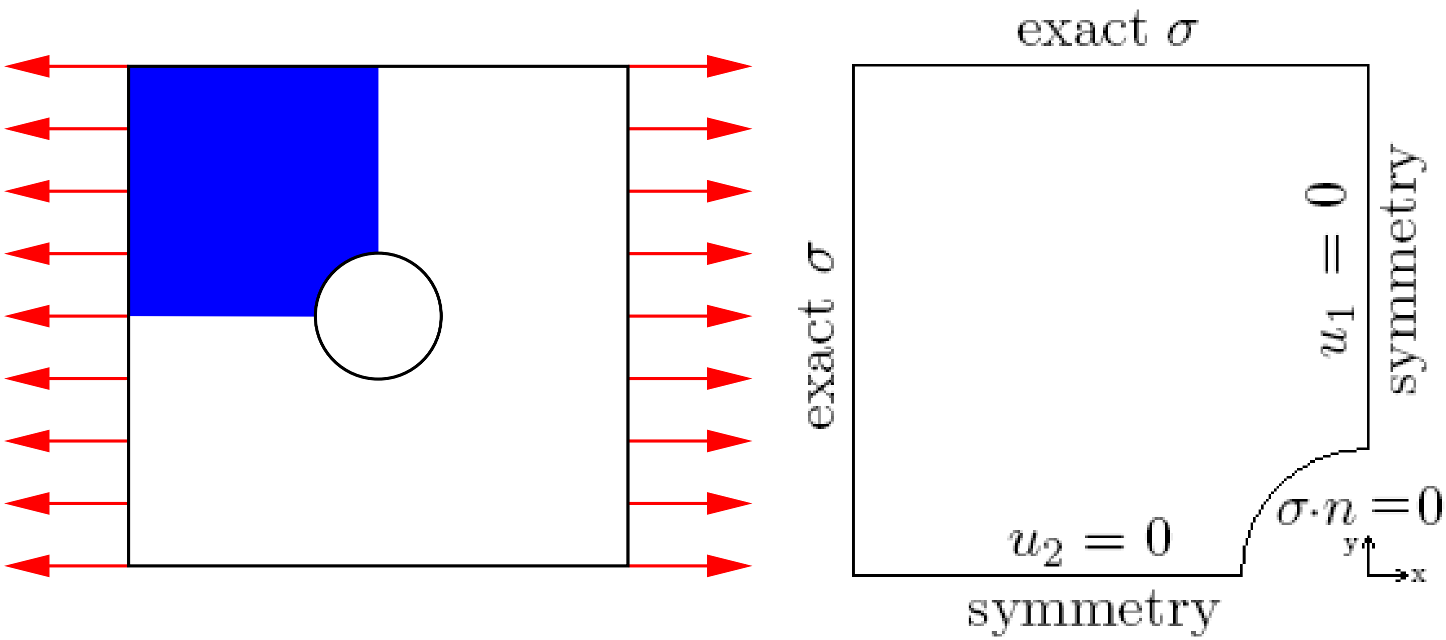
Example 1: Stationary heat conduction



The use of T-splines leads to a significant improvement.

Example 2: Linear Elasticity

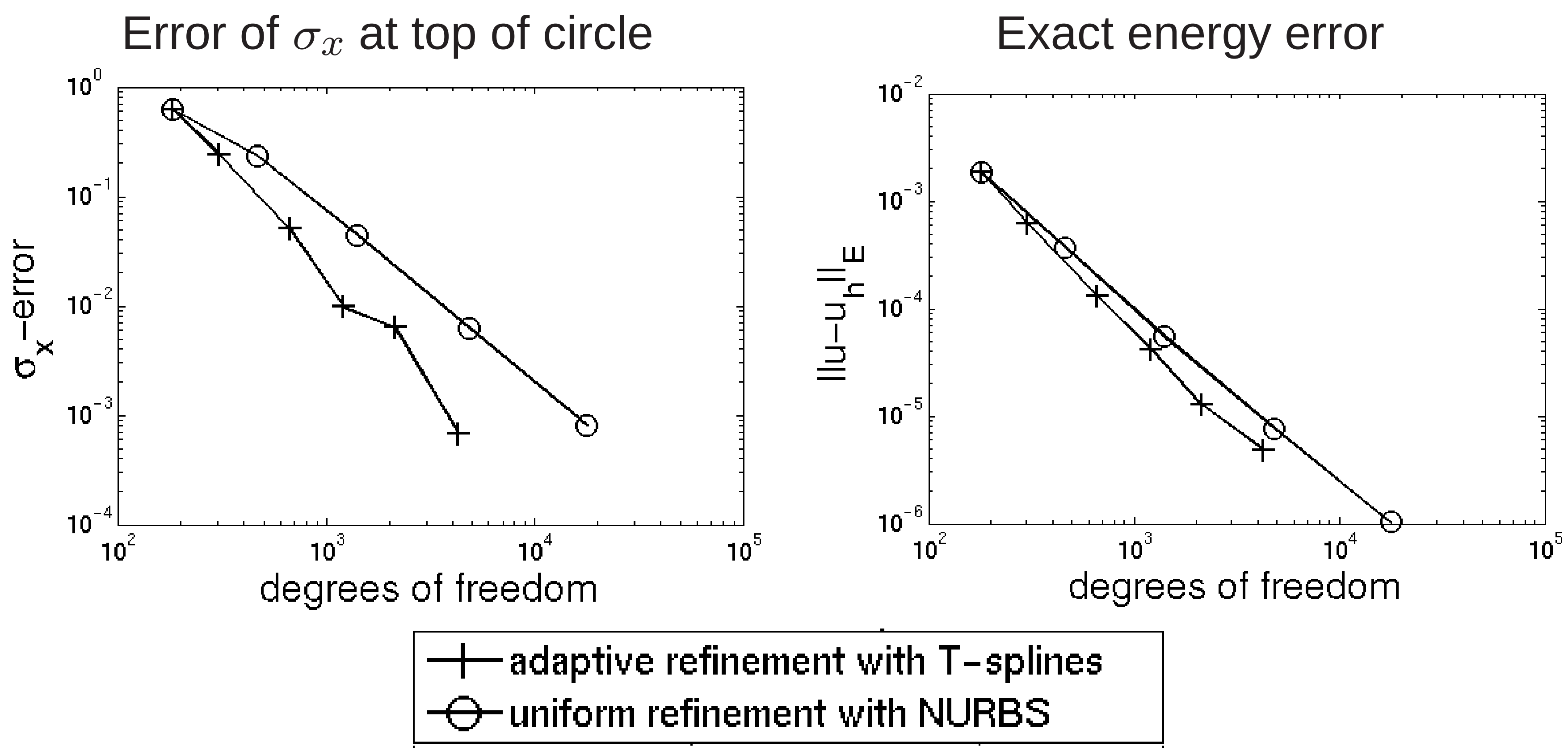
Solve the Laplace equation div$\sigma(u) = 0$



subject to Dirichlet and von Neumann boundary conditions derived from the exact solution, which is known for a homogeneous and isotropic material.

description of the domain: global C^0 parameterization without singular points

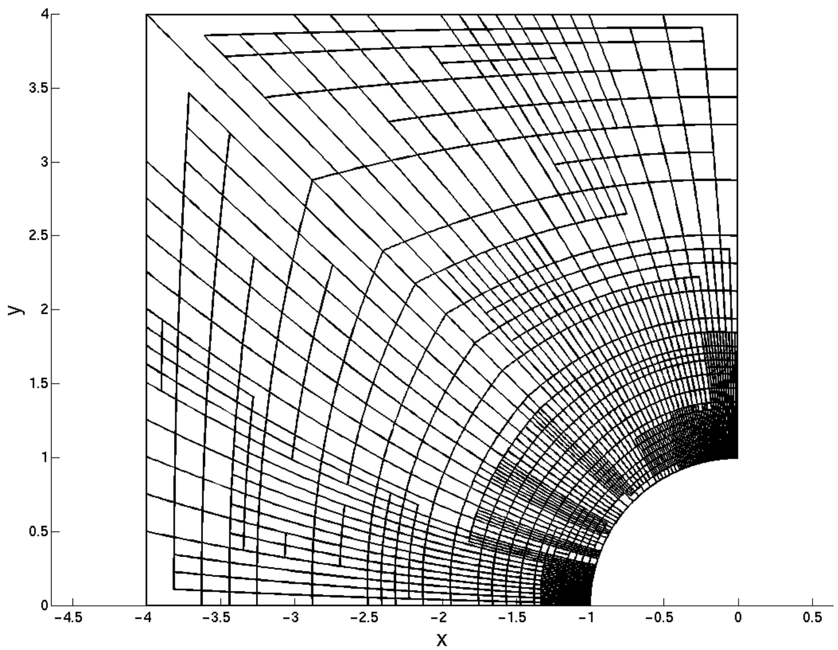
Example 2: Linear Elasticity



The use of T-splines leads to a slight improvement.

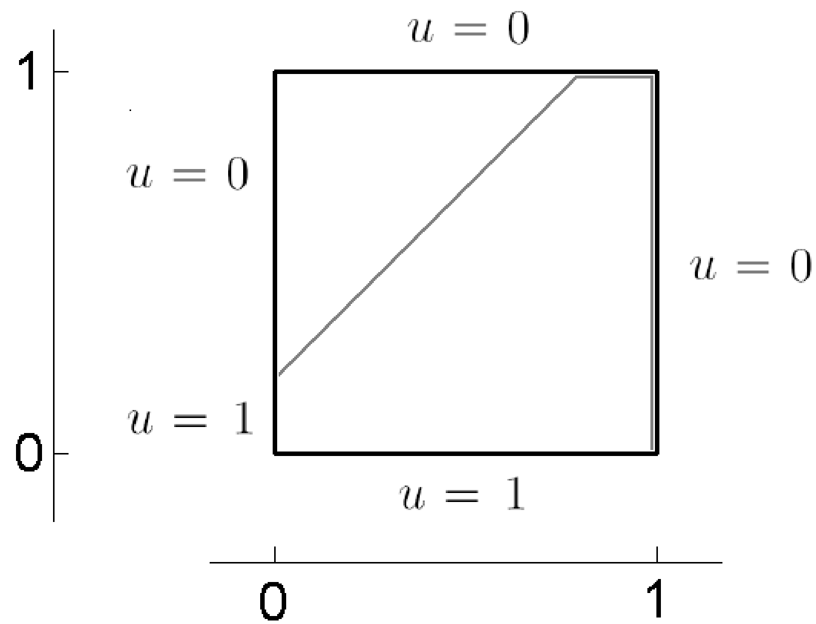
Example 2: Linear Elasticity

The refined T-mesh (mapped onto Ω) after 5 refinement steps, 4302 dof.



Example 3: Advection Dominated Advection–Diffusion

Solve $\kappa\Delta u + \mathbf{a}\cdot\nabla u = 0$ with diffusion coefficient $\kappa = 10^{-6}$ and advection velocity $\mathbf{a} = (\sin\theta, \cos\theta)$ for $\theta = 45°$.

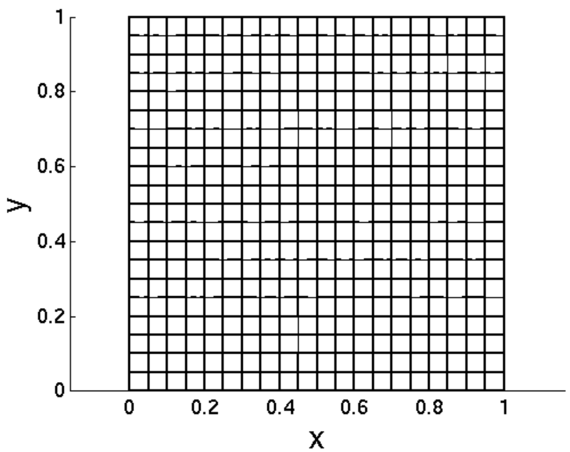


grey: estimated position of sharp layers

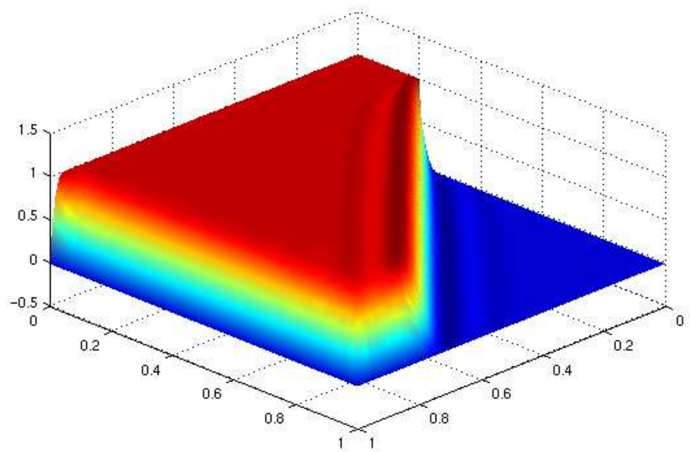
is solved using SUPG stabilization

Example 3: Advection Dominated Advection–Diffusion

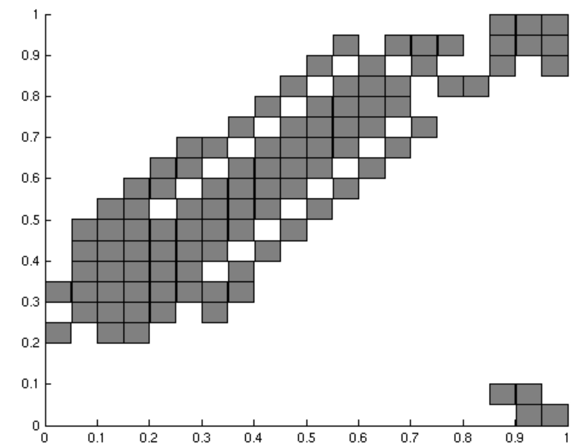
T-mesh



Solution

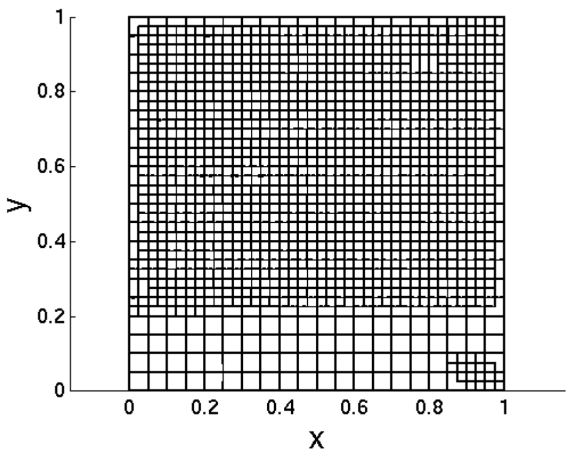


patches marked for refinement

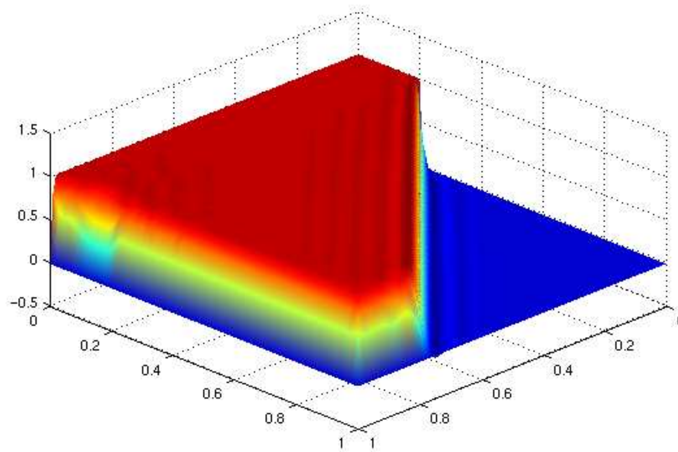


Example 3: Advection Dominated Advection–Diffusion

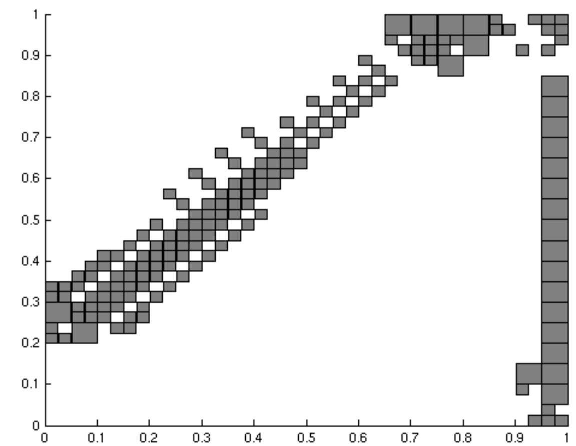
T-mesh



Solution

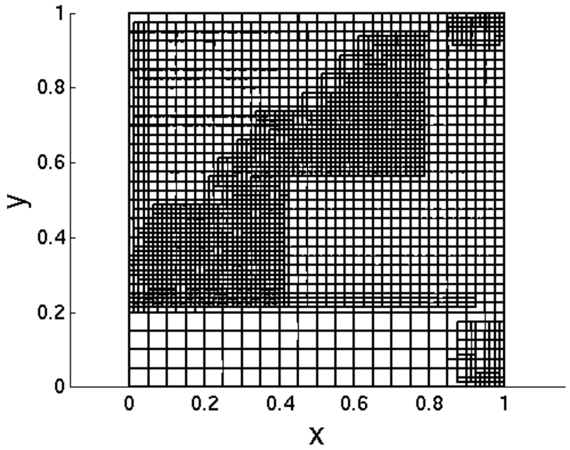


patches marked for refinement

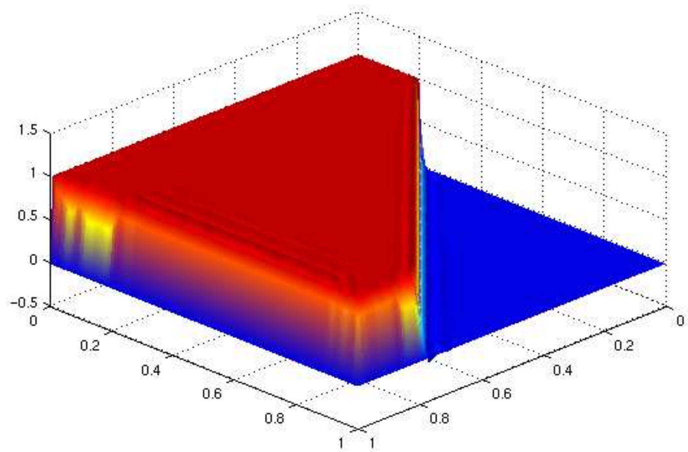


Example 3: Advection Dominated Advection–Diffusion

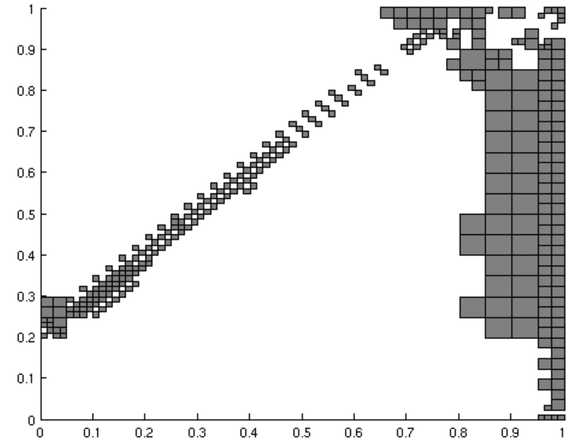
T-mesh



Solution

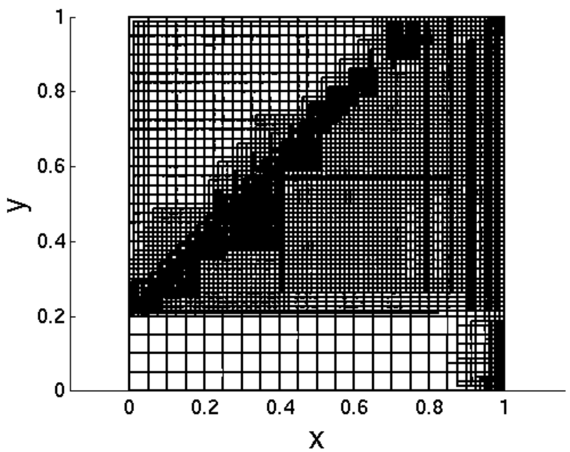


patches marked for refinement

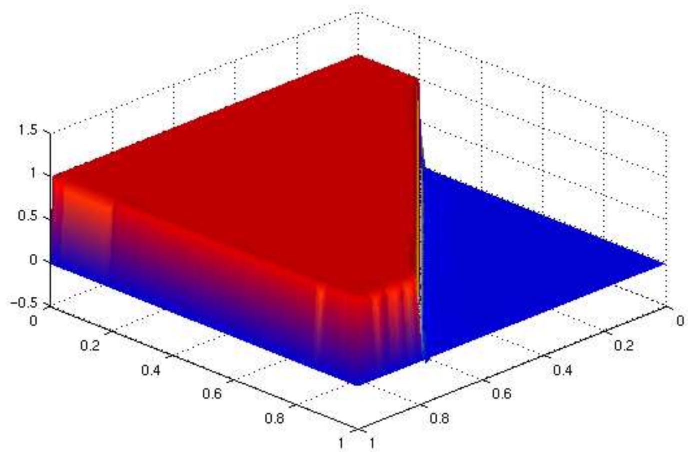


Example 3: Advection Dominated Advection–Diffusion

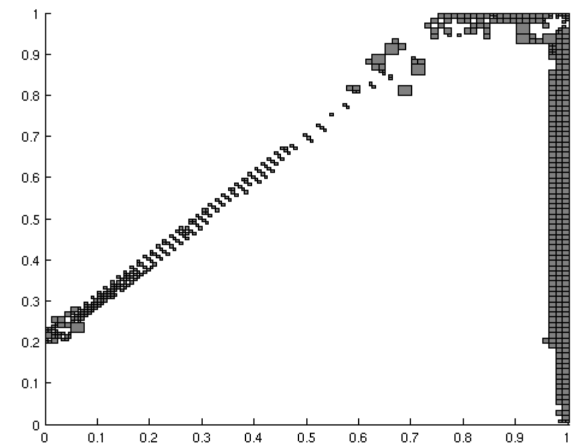
T-mesh



Solution

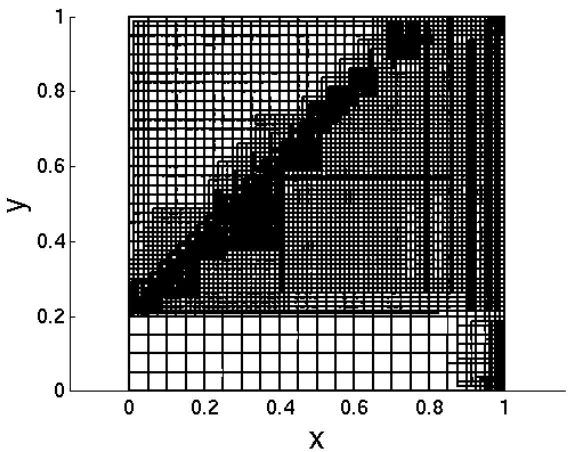


patches marked for refinement

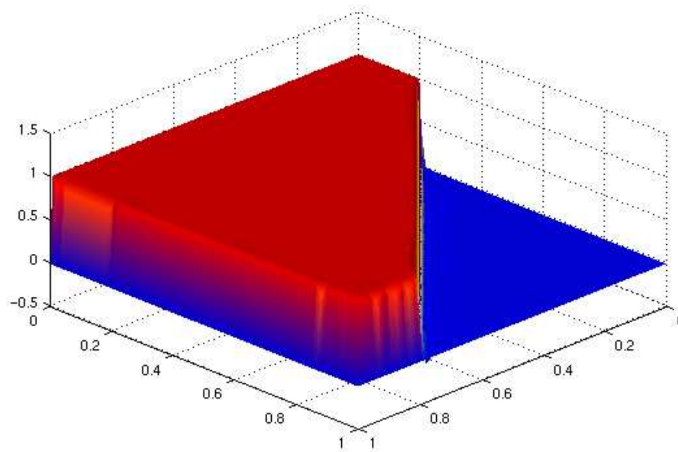


Example 3: Advection Dominated Advection–Diffusion

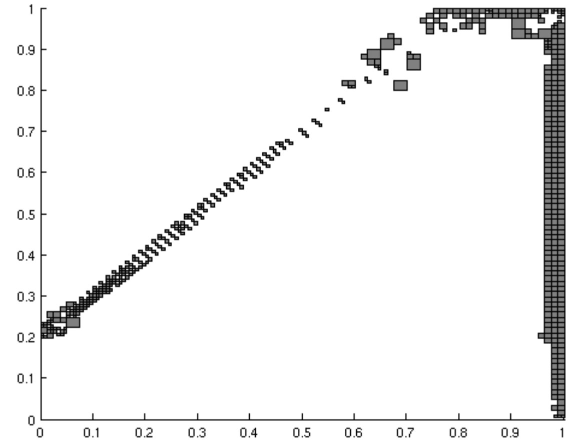
T-mesh



Solution



patches marked for refinement

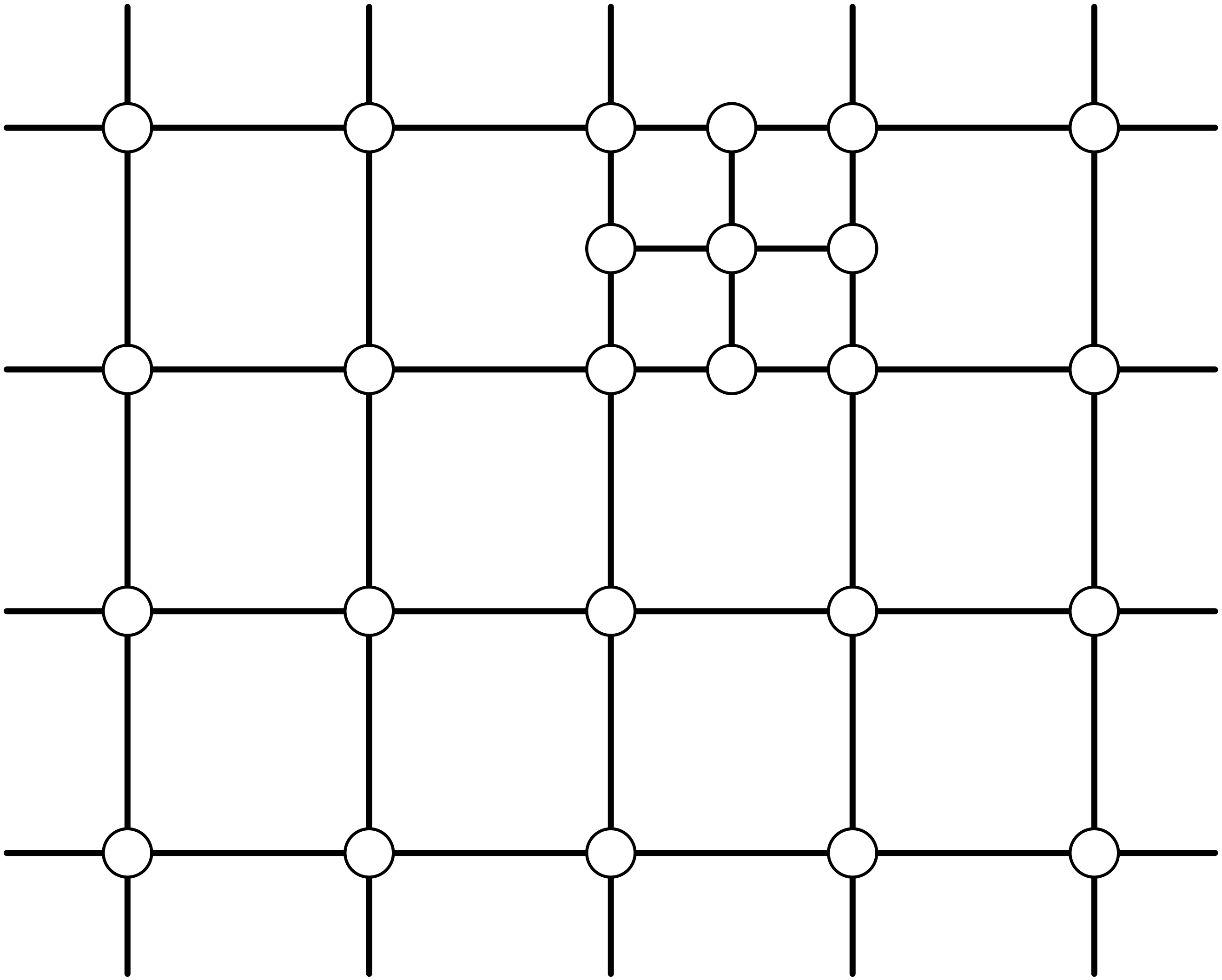


Refinement of T-splines is not as local as we hoped it to be!

Insertion of a grid point may trigger a chain of additional grid point insertions, in order to get a refinement of the previous T-spline space.

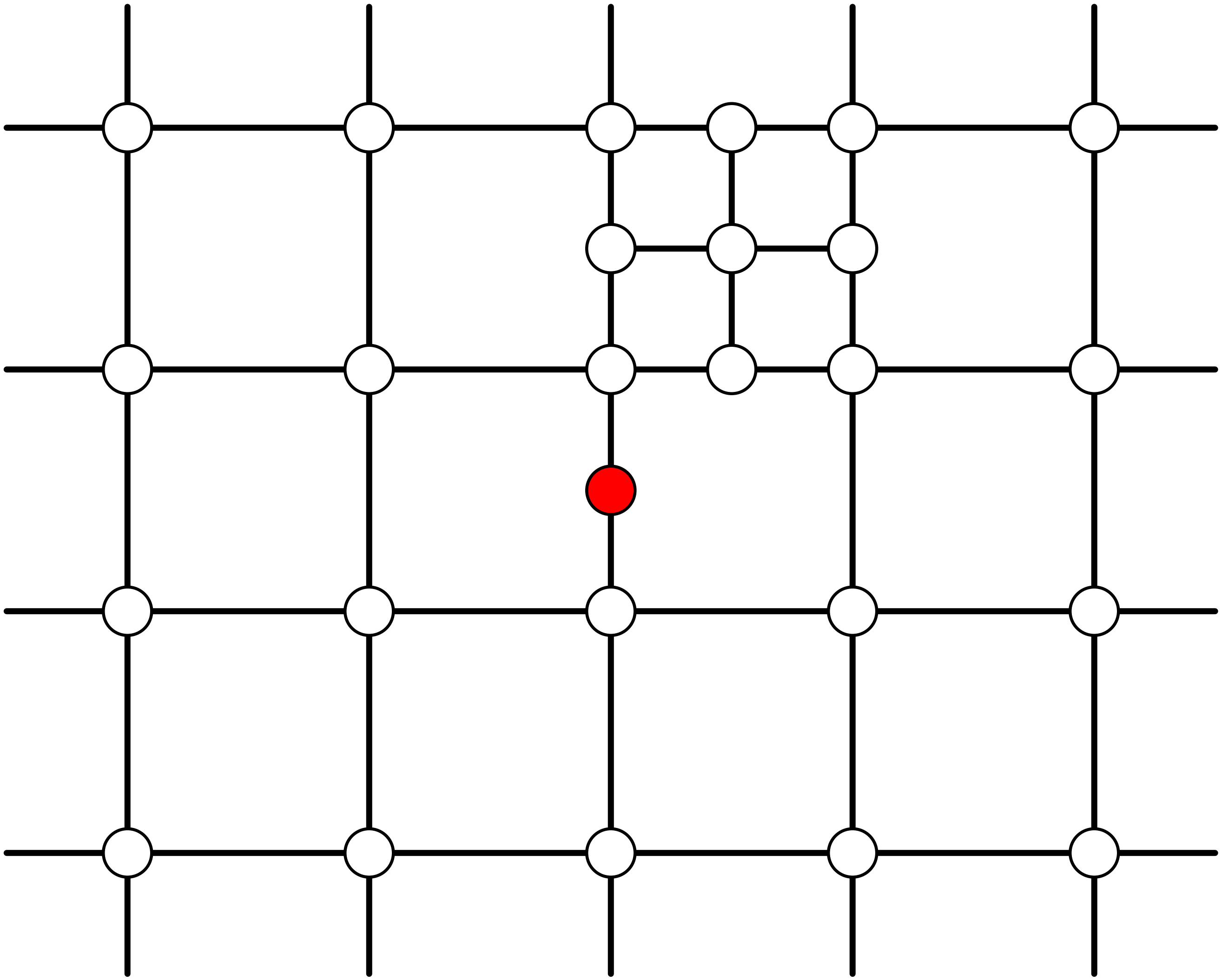
This seems to be especially worse for refinement along diagonals.

Effect of grid point insertions



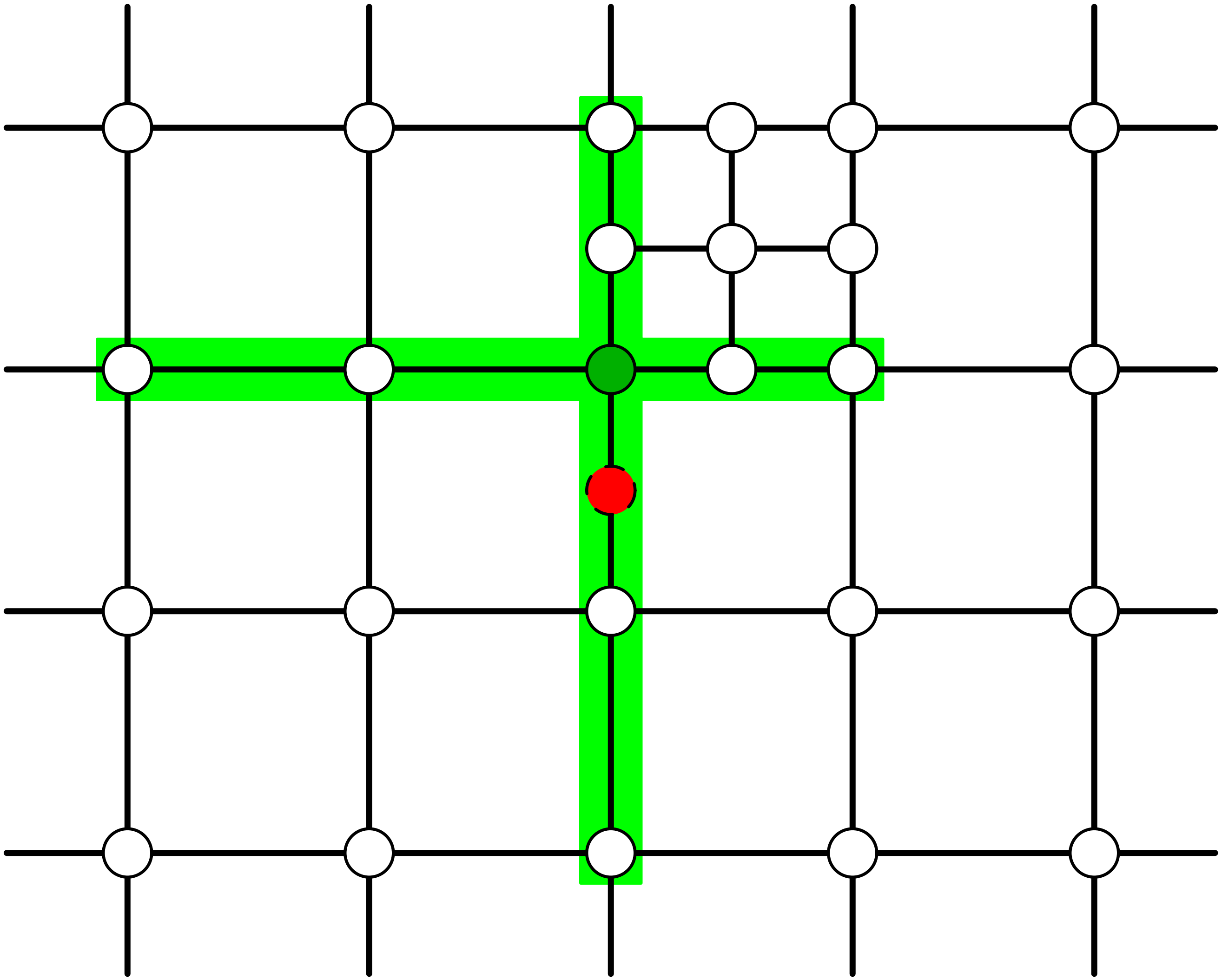
Original T-spline grid

Effect of grid point insertions



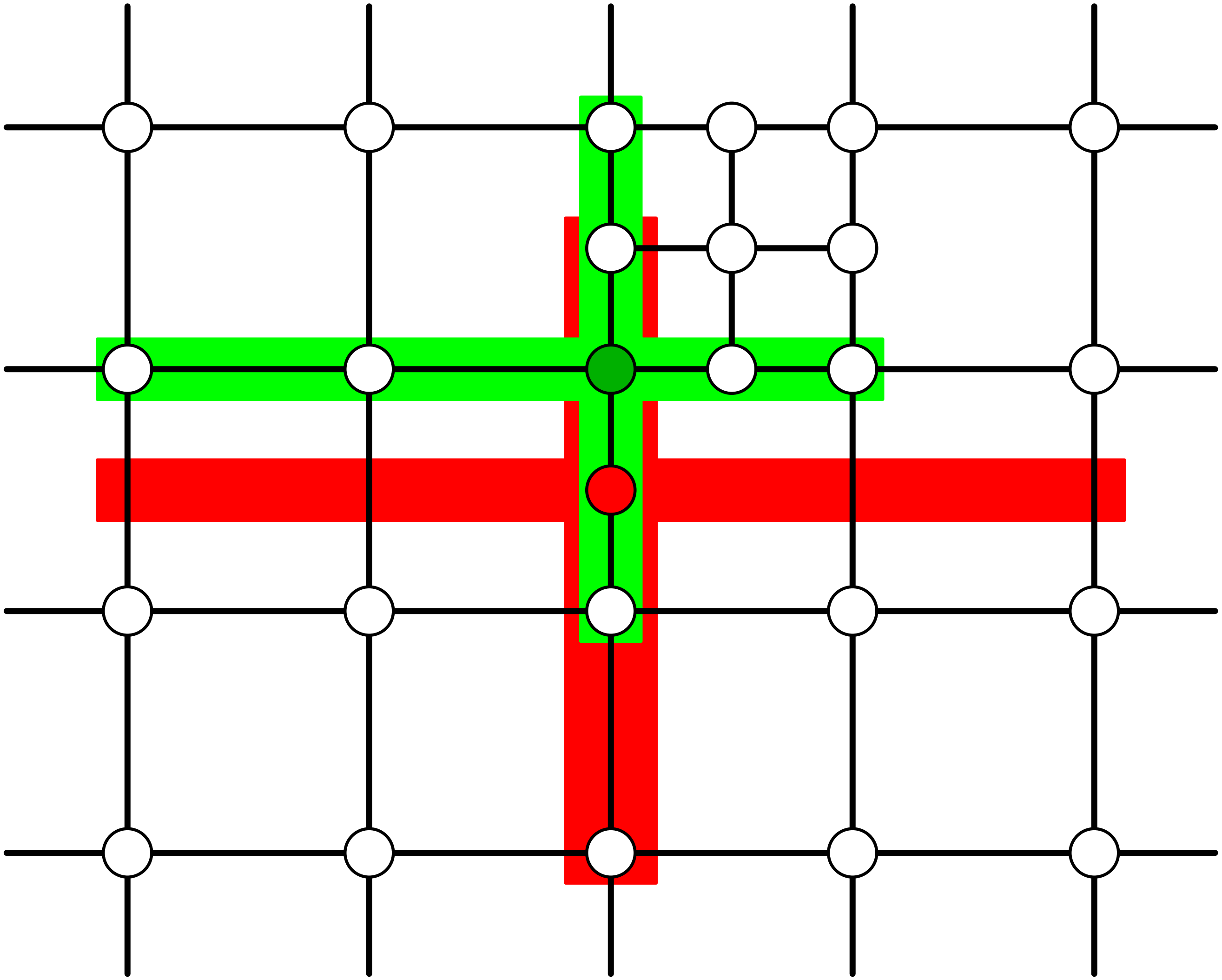
One grid point is to be inserted

Effect of grid point insertions



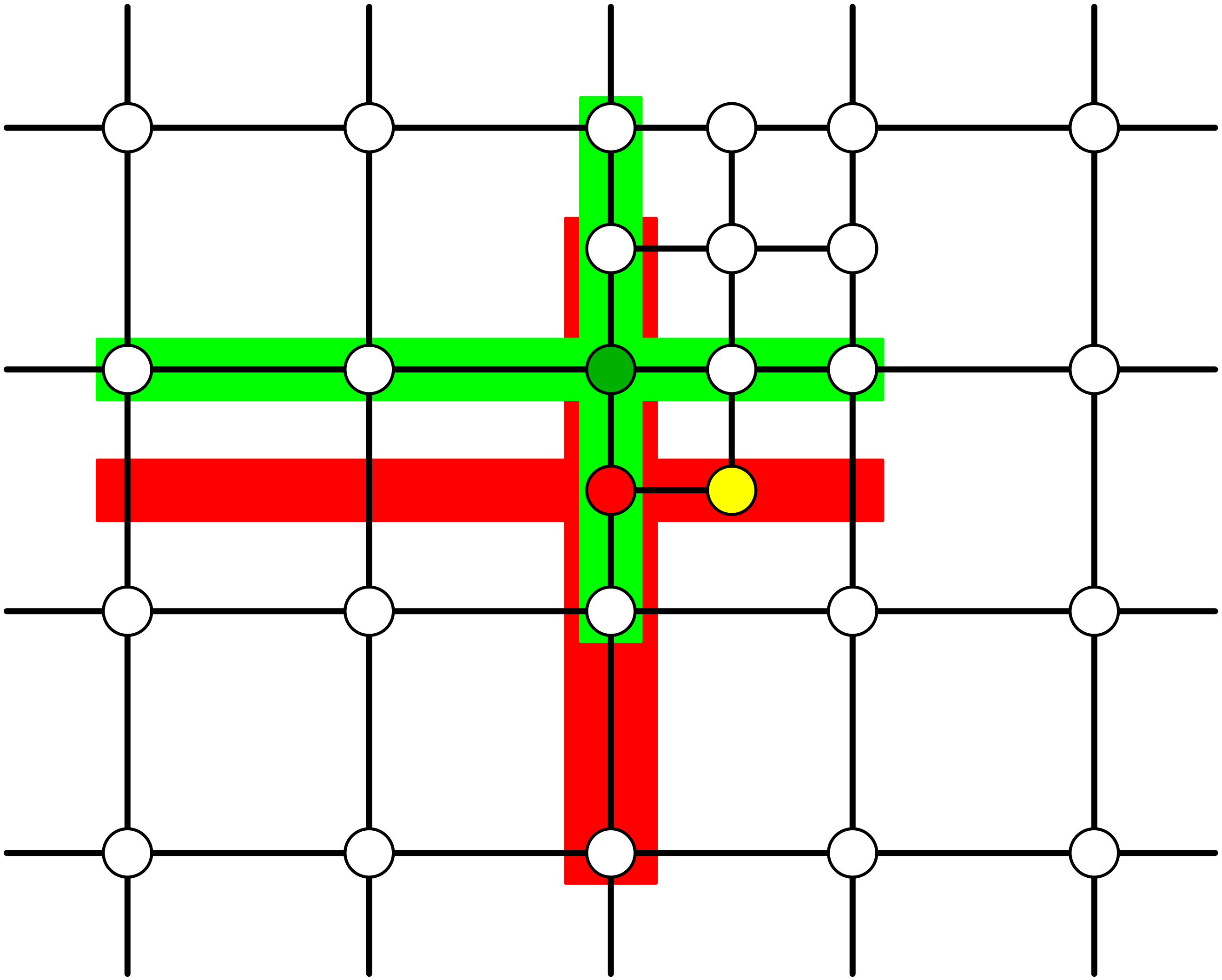
This blending function is affected.

Effect of grid point insertions



It is to be split into two blending functions.

Effect of grid point insertions



This requires **another grid point.**

Outline

- Preliminaries: Galerkin projection, Isogeometric approach
- Tensor-product splines and T-splines
- Examples
- **Future Work: EXCITING**

EXCITING Future Work

EXCITING is a project in FP7 of the EU, program SST (sustainable surface transportation), 2008-2011, negotiation pending.

Exact Geometry Simulation for Optimized Design of Vehicles and Vessels

“EGSODVV”

EXCITING Future Work

EXCITING is a project in FP7 of the EU, program SST (sustainable surface transportation), 2008-2011, negotiation pending.

Exact Geometry Simulation for Optimized Design of Vehicles and Vessels

“EXCITING”

EXCITING Future Work

EXCITING is a project in FP7 of the EU, program SST (sustainable surface transportation), 2008-2011, negotiation pending.

Exact Geometry Simulation for Optimized Design of Vehicles and Vessels

“EXCITING”

We will apply **isogeometric analysis** to functional free-form surfaces and core components of vehicles and vessels: **ship hulls, ship propellers, car components and frames.**

The structure of EXCITING

JKU Linz / B. Jüttler

VA Tech HYDRO (ship propellers)

TU Munich / B. Simeon

SIEMENS (car components)

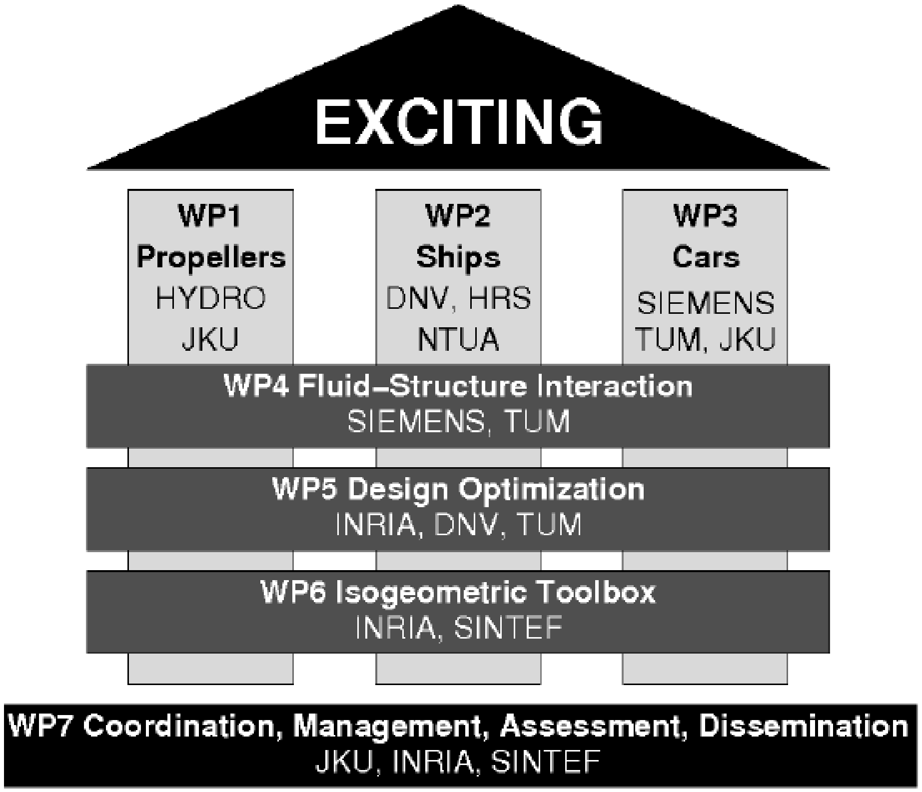
NTU Athens / P. Kaklis

HRS (ships)

SINTEF / T. Dokken

DNV (ships)

INRIA / B. Mourrain



EXCITING challenges

Eliminate mesh generation. All computational tools will be based on the same representation of geometry.

Fluid structure interaction. Isogeometric solver with exact description of the interface.

Automated design optimization. Build design optimization loop based on iso-geometric numerical simulation.

Isogeometric toolbox. Make the specific tools (for propellers, ship hulls, car components) which will be developed in the project useful for a wider range of problems.