Steinitz's Theorem Project Report

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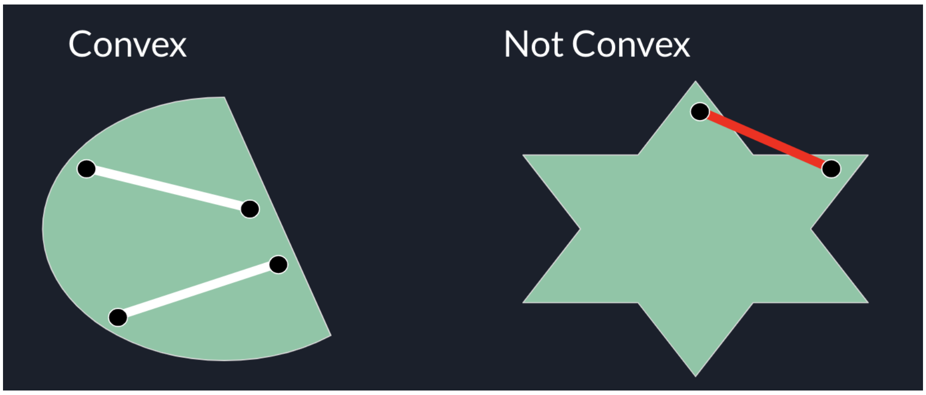
§1 Introduction

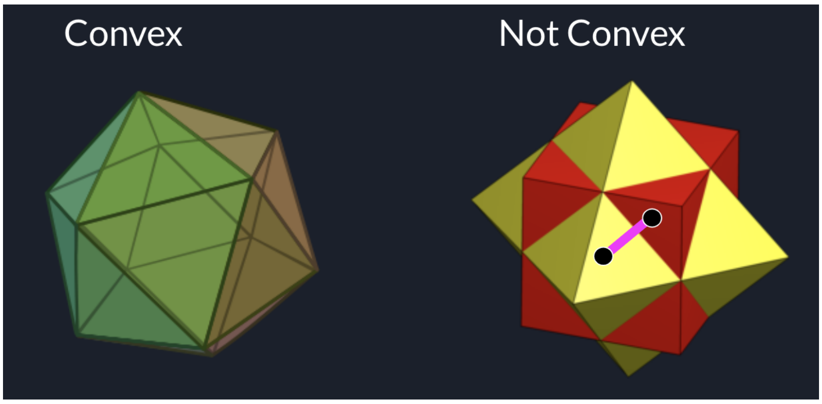
Looking at the vertices and edges of polyhedra gives a family of graphs that we might expect has nice properties. It turns out that there is actually a very nice characterization of these graphs! We can use this characterization to find useful representations of certain graphs.

§2 Basic Definitions

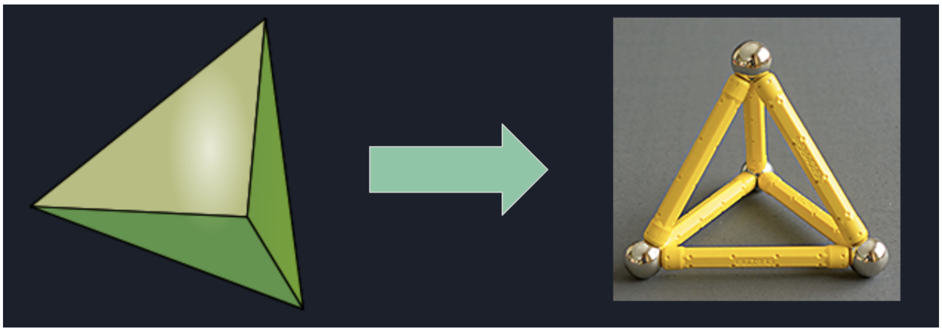
We define a space to be **convex** if the line segment connecting any two points in the space remains entirely inside the space. This works for two-dimensional sets:



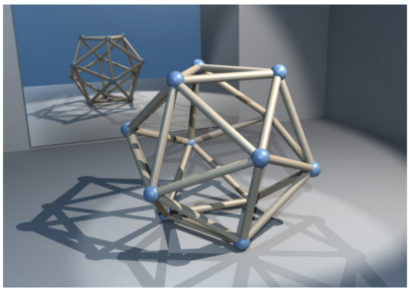
and three-dimensional sets:

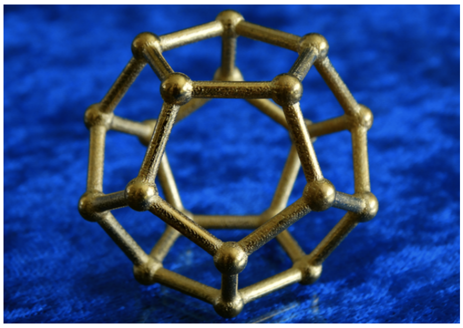


Given a polyhedron, we define its **1-skeleton** to be the graph formed from the vertices and edges of the polyhedron. For example, the 1-skeleton of a tetrahedron is K_4:



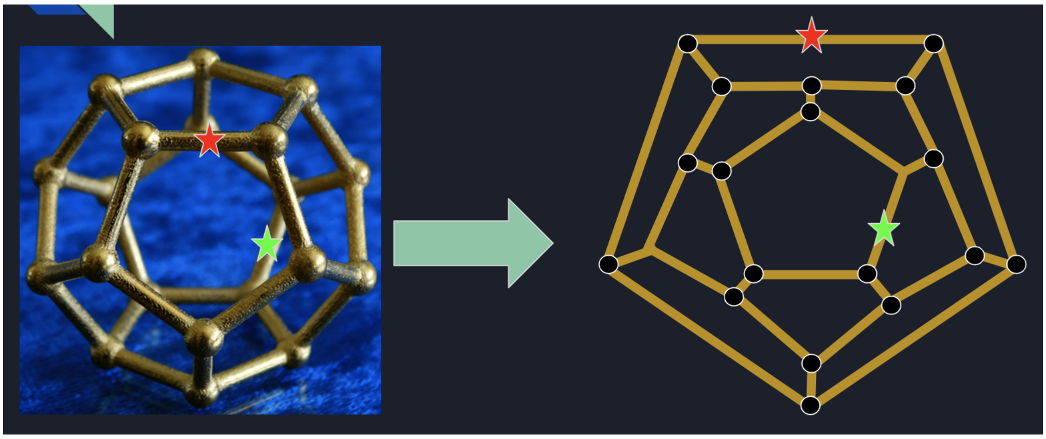
Here are some further examples of the 1-skeleton of an icosahedron and a dodecahedron:





§3 Properties of 1-Skeletons

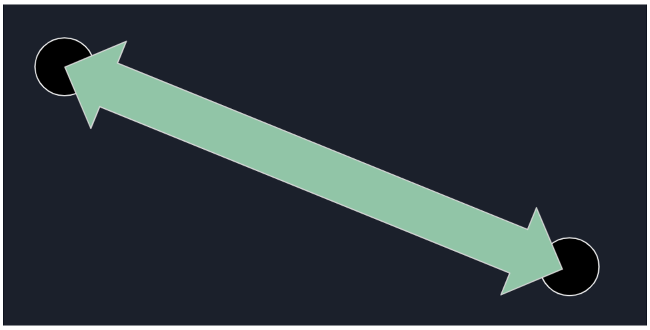
What properties do we know the 1-skeleton of a convex polyhedron must have? First, it must be planar. To see this, imagine moving your eye towards one of the faces until you are close enough that all of the other faces appear "inside" the face you are looking through, as shown here:



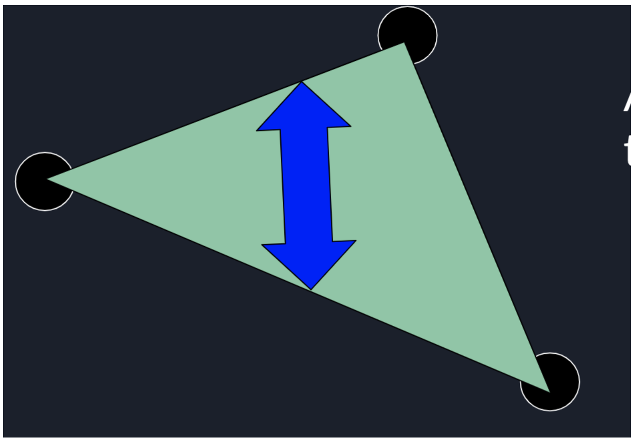
This is always possible because the polyedron is convex, meaning intuitively it doesn't have any parts that "jut out". The graph formed from viewing in this way will have no intersections because the polyhedron is convex, so the straight-line rays our eyes see are not allowed to leave via an edge on the boundary of the polyhedron and then go back inside. As a side note, we could actually calculate a suitable place to view the face from since any convex polyhedron is just the intersection of the "half-spaces" (all of space that is on one specified side of a plane) of the planes constituting its faces, and so we could choose a point closer to the polyhedron that the intersections of these planes (specifically the intersection of the planes corresponding to faces adjacent to the one we are looking through).

The 1-skeleton of a polyhedron is also what we call three-connected. A graph is defined to be **three-connected** if the graph has more than three vertices and the removal of any two vertices (and their incident edges) leaves the graph connected. Why must a 1-skeleton have more than three vertices? Well, if we have just one vertex the only convex shape we can make is a single point:

If we allow two vertices, the only convex possibility is a line segment because intuitively if we include any points outside of the line segment, we must have all of the points connecting that point to the line segment, so we would get a triangle and therefore another vertex:

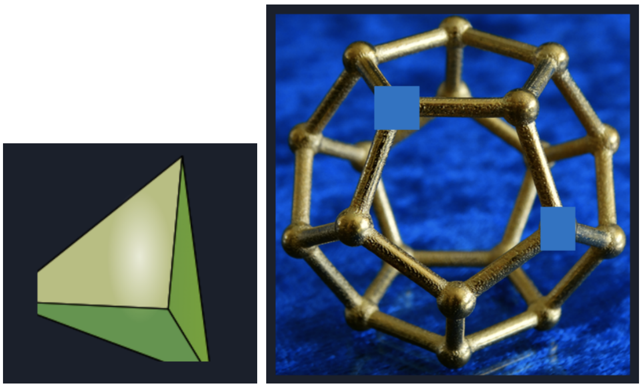


Similarly, if we only have three vertices, we can connect all of the line segments between them and then connect those line segments with all possible segments, to obtain a "maximal" convex figure of a triangle:



More formally, we can say that the convex hull of three points is only a triangle or some degenerate shape, and since a convex polytope is the convex hull of its vertices we know that we must have at least four vertices to obtain a polyhedron.

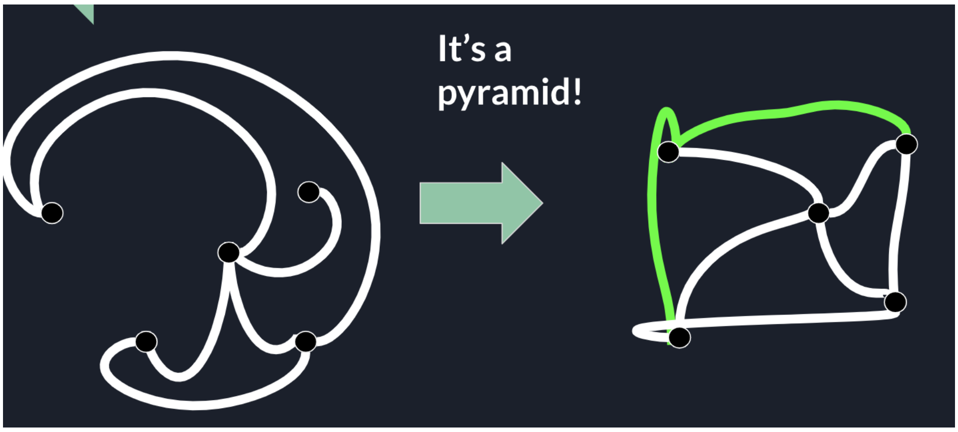
Lastly, we need to show that a 1-skeleton stays connected upon the removal of any two vertices. This is difficult to show formally, but intuitively we can get a sense that it might be true because each vertex must have degree at least three (otherwise the vertex would be "flat", so there could be no vertex), and so it makes sense that there are at least three completely distinct paths between any two vertices. We can see by example that the removal of two vertices of a tetrahedron, which we would think is approximately the "minimally connected" polyhedron, leaves it connected and it also works for the dodecahedron:



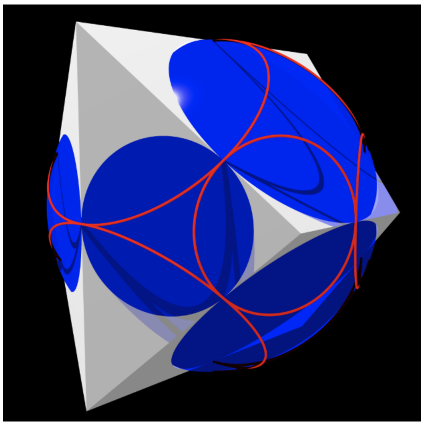
§4 Steinitz's Theorem and Applications

Rather surprisingly, **Steinitz's Theorem** says that any graph which is planar and three-connected is the 1-skeleton of some convex polyhedron! This result lends itself immediately to two useful consequences.

The first of these is that any planar graph can be drawn planarly using only straight-line edges. The main idea behind this is that we can always add more vertices and edges to our graph in a planar way until it is three-connected. Once it is three-connected, we know it comes from the 1-skeleton of some polyhedron and so when we look through a face of that polyhedron, we get a drawing that uses only straight lines. To conclude, we then just remove the extra vertices and edges we have added. A formal proof that we can always add vertices and edges to obtain a three-connected planar graph is beyond the scope of this paper.



Another interesting application of Steinitz's Theorem is the ability to represent any three-connected planar graph as a network of tangent circles, where the centers of the circles are the vertices and the points of tangency are the edges. This works because any convex polyhedron (if we are allowed to resize certain edges) has a **midsphere**, a sphere tangent to every edge. Therefore, when we find draw the intersection of the midsphere with the polyhedron corresponding to our graph, we obtain the desired network of circles:



§5 Conclusion

The 1-skeletons of polyhedra are exactly the graphs which are planar and three-connected, which is known as Steinitz's Theorem. We can use this fact to find geometrically useful representations of graphs, both as straight-line embeddings and as a sort of circle-packing representation. Both of these are related to minimal-energy representations of graphs, which can be of use to both scientists and geometers alike.

§6 References

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Images

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