Generalized Euler-Poincaré Theorem

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The Euler-Poincaré Theorem relates the numbers of vertices, V, edges E, faces F, cells C, etc, of graphs, polygons, polyhedra, and even higher-dimensional polytopes. It can be presented in many different ways. For a single 3-dimensional polyhedral body without any holes, Euler [1] originally stated it as:

$$V + F = E + 2. \tag{1}$$

Poincaré [2] extended the formulation to such a body in D-dimensional space:

$$N_0 - N_1 + N_2 - N_3 + \ldots N_{D-1} = 1 + (-1)^{D-1}. \tag{2}$$

Here N_i denotes an element of dimensionality i; e.g., N_0 represents the number of vertices. For D = 3 this formula reduces to Eqn.1.

In my work in computer-aided design and solid modeling, I use this formula to check the consistency and validity of the models I want to build on rapid-prototyping machines. However, I normally deal with more complex objects that also may have holes or tunnels, and sometimes the data file contains a description of several objects. Thus I need a more general formula that can accommodate all these cases. I like the following form:

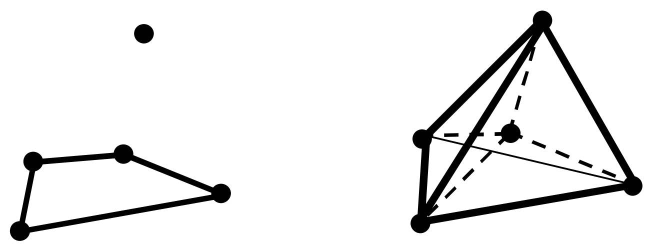
$$I - N_0 + N_1 - N_2 + N_3 - \ldots N_D = R_1 - R_2 + R_3 - \ldots R_D. \tag{3}$$

On the left-hand side is again Poincaré's list of i-dimensional building elements; however, the list has been extended by one additional term on either end: I denotes the number of individual, non-connected assemblies of such components; and the count of the building blocks now includes elements of the same dimension as the dimension in which the assembly is embedded. Note that the inclusion of these two terms allows us to get rid of the less than elegant term $1 + (-1)^{D-1}$ in Eqn.2.

On the right-hand side, we tally all the elements of various dimensions that form closed rings in and by themselves. Thus R_1 counts closed-ring edges; R_2 annular, ring-shaped faces; and R_3 solid-body handles. The exact use of these elements will become clear when we discuss an inductive construction of this formula.

Euler's theorem can be proven in many different ways. Eppstein cites 17 different proofs [3]. An instructive and inductive proof can be obtained by a simple counting process, applied while one incrementally constructs the object of interest. For instance, let's start with *v* isolated vertices (N_0) in a plane. These obviously form *v* separate, individual components (I); thus only the first two terms appear in Eqn.3, and the equation trivially balances. As we start adding edges between pairs of vertices, we add terms of the third kind (N_1); but for each such connection made, the number of individual components is lowered by one, and the equation thus stays balanced.

When we close a sequence of edges into a cycle, we gain an edge without further reducing the number of connected components. However, we also gain a loop (N_2), which we must count as a new, separate entity (Fig.1a). It is counted as an element of dimensionality 2, because we can stretch a "membrane" over the area encircled by the loop, and make this into a face. Counting these loop/face elements, formed by cycles with an empty interior, keeps the equation balanced.



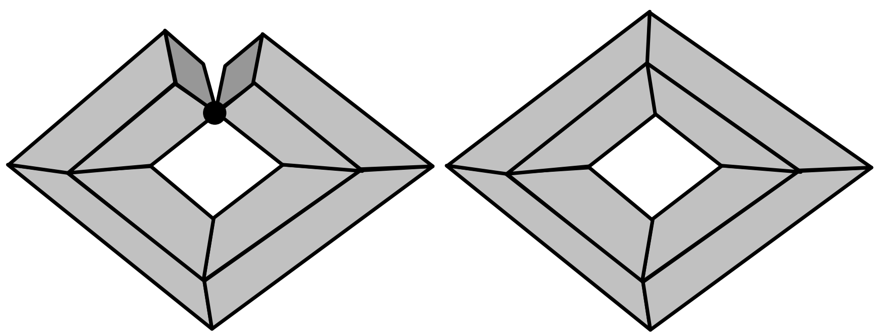
(a) $2 - 5 + 4 - 1 = 0$.   **(b)** $1 - 5 + 10 - 10 + 5 - 1 = 0$.

Figure 1: An incremental construction. (a) Five partially connected points in 2D; (b) the 4-dimensional simplex featuring five 3D solid cells (N_3), and one 4D hypercell (N_4). In both cases, the relevant values inserted into Eqn.3 are also shown.

A minimal cycle can be formed with a single vertex and a single edge returning to that same vertex. But what happens, if we remove the vertex and just consider a single edge forming a ring? Such a ring-edge is a special topological element indeed, and it is counted separately as R_1 on the right hand side of Eqn.3.

Cycles of either kind also allows us to draw internal contours inside a face and then remove the membrane inside this inner contour, thus opening up a hole in the face. If the face surrounding the hole is subdivided by some edges so that the inner cyclic contour is edge-connected to the outer contour, then Eqn.3 will readily balance. But if we remove the last connecting edge between the inner and outer contours, we need a new term to balance that loss, since the number of individual components (I) does not change, as long as the two contours are connected by a face. However, if this is the only connection between the two contours, then this is a very special face, since it forms a ring or annulus all by itself. This new topological entity is counted with the term R_2 in Eqn.3. Thus for every hole in a face surrounded by an undivided annulus, we have to increment the ring-face term (R_2).

Let's assume we continue to create many new loops and turn them into faces as soon as we add the closing edge on a cycle. But now let's take our construction out of the plane and contemplate it in 3D space. We can then bend our collection of faces into a bowl and aim at eventually closing it into an orientable (two-sided) shell without any holes. Topologically, something new happens again, when we turn the last cycle into a face and thus close the last opening in the shell: At this moment, we isolate a piece of 3D space and created a separate 3D cell (N_3). Counting these cells with term N_3, balances the face (N_2) that we gained "for free" by filling in a cycle that we had not previously counted because it was the outer surrounding contour of a graph in a plane. Eqn.3 continues to work, as we create clusters of adjacent cells, as long as we count individually all cells that are separated from one another by a membrane of faces. Thus we can now handle 3D objects with vertices, edges, faces, and cells. Of course, cells could be "filled in" and thereby be turned into solid bodies.



(a) 1 – 19 + 36 – (**1**+18) + 1 = 0. **(b)** 1 – 16 + 32 – (**1**+16) + 1 = 0 – 0 + 1.

Figure 2: Creating a solid handle: (a) Situation when a "worm" is bent into a loop and creates a cycle (**1**); and (b) after the two end-faces have been merged into a single internal membrane and then removed to form a cyclic ring-shaped handle.

Something special happens again, if we create a torus, i.e., a shell in the shape of a doughnut or in the shape of a handle on some other solid object. We can create such a closed solid handle from a cylindrical worm by fusing its two end-faces. In so doing, we loose these two end-faces, but what do we gain in turn? First we gain a very visible ring, which somehow divides space in such a way that we can distinguish between infinitely long threads that go through this ring and those that don't; this happens when we first merge two vertices of the two end-faces (Fig.2a). This central cycle (which could also be turned into a face) is counted in term N_2. Topologically the situation remains the same while we completely fuse the two end-faces but retain them as an inner membrane. But when we then remove this shared inner face, we also change the nature of the space internal to the torus: We can now go around the inside of this loop without having to cross any faces. This new ring cell is accounted for in term R_3. Thus for each solid handle that we add to a solid object, we gain two loops: One is formed by the handle surface (N_2), and one is internal to the handle body (R_3). The number of such handles on an object is also called its genus; i.e., a sphere has genus zero, and a donut has genus 1.

When we start to look at a cluster of solid cells from 4-dimensional space, we can conceive of bending that cluster into a curved, and possibly closed, hyper-shell (N_4), in analogy to the way that we took an originally planar collection of vertices, edges, and faces, and then considered it to be a partial 3D shell of a solid object. Similarly, when we now plug the last hole in a 4D hyper-shell composed of 3D solid cells, we gain an additional 3D solid without having to "expend" any new vertices, edges or faces, – it is just a matter how we colour that region of space – either as a hole or as a solid piece. To keep the equation balanced, we now need to count the newly generated hyper-cell (N_4) with a sign opposite to that of the 3D cells (N_3).

You may suspect, that tricky issues arise as we contemplate higher-dimensional loops and handles, – but this is beyond the scope of this little treatise of the Euler-Poincaré equation. Eqn.3 should be good enough for use on a deserted island on a 3D world.

References

[1] L. Euler, "Elementa doctrinae solidorum -- Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita," Novi comment acad. sc. imp. Petropol., 4, 1752-3, 109-140-160.

[2] H. Poincaré, "Analysis Situs," Jour. École Polytechnique 2, 1 (1895).

[3] D. Eppstein, "Seventeen Proofs of Euler's Formula: V–E+F=2," http://www1.ics.uci.edu/~eppstein/junkyard/euler/