Escher Sphere Construction Kit

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ABSTRACT

M.C. Escher created a myriad of amazing planar tessellations, yet only a few three-dimensional ones such as his wooden fish ball and dodecahedral flower. We have developed an interactive program to design and manufacture "Escher Spheres" - sets of tiles that can be assembled into spherical balls. The user chooses from a set of predefined symmetry groups and then deforms the boundaries of the basic domain tile; all corresponding points based on the chosen symmetry class move concurrently, instantly showing the overall result. The interior of the tile can be embellished with a bas-relief. Finally the tile is radially extruded and output as a solid model suitable for free-form fabrication.

Categories and Subject Descriptors

J.6 [**Computer Appl.**]: Computer-Aided Engineering

General Terms

Computer-aided design (CAD)

Keywords

M.C. Escher, solid modeling, tessellation, spherical symmetry, spherical tiling, tile editor

1. INTRODUCTION

M.C. Escher is arguably the most famous graphic artist of the 20th century, celebrated for his artistic vision of mathematics [5]. In particular Escher created a spectacular array of drawings, tiling the plane with such creatures as lizards, birds, and fish. Escher himself found this subject the most interesting of all his work [3] and used his two-dimensional drawings as the basis for his hobby of carving beechwood spheres [12]. Such sculptures include a sphere of intertwined fish and a dodecahedral flower (Fig. 1). In contrast to the 137 regular divisions of the plane that he created, Escher only made a handful of spherically symmetric sculptures. There are several reasons for this disparity.

Figure 1: Sculptures carved by M.C. Escher.

Trained as a print maker, Escher could make a sketch, turn it into an etching, and easily make reproductions. However, creating spherically symmetric sculptures is much more complex. First, it is difficult to understand how spherical symmetries work to form a whole object - how the pieces join together. Second, it is much harder to visualize a spherical design than a planar tiling. Escher could not simply make sketches but had to use special devices and models made of wood and thick paper [3]. Finally, not only was the carving of an original a lengthy process, but making reproductions was difficult and expensive.

During the last decade, many Solid Free-form Fabrication (SFF) processes have emerged making it easy - although not inexpensive - to build shapes of almost arbitrary complex geometry. This leaves the design of such objects as the dominant bottleneck. We have thus set out to develop an interactive system to easily design and manufacture "Escher Spheres" - spherical balls composed of identical tiles. Our system offers interactive editing capabilities for modifying the shape of the tile, embellishing it with a bas-relief, and extruding it into a manufacturable physical part. The key challenge was how to hide the complexities of spherical tilings and to make the design process manageable for as large an audience as possible.

2. SPHERICAL TESSELLATIONS

A tessellation is a regular tiling, or repeating pattern, that fills a surface without gaps or overlaps. Any tile can be transformed by a symmetry operation to another tile of the tessellation. The set of all such operations of a tessellation forms a symmetry group. To understand how a tessellation is created, we first review planar tilings.

2.1 Planar Tiling

To create an artistic planar tiling, we start with a basic shape that tiles the plane - equilateral triangle, square, hexagon, etc. This tile can now be modified into an interesting figurative shape. The underlying symmetry of the chosen tessellation must be maintained, so that the modified tiles will still fit together seamlessly and cover the entire plane. For example, when a "bulge" is made on one side of the basic shape, a corresponding bulge may have to be taken away from the opposite side [13][9][1]. Editing systems for such planar tilings are available commercially as well as on the web [15][10]. Interesting work has also been done in the "Escherization" of images [11].

2.2 Spherical Tiling

As with planar tiling, in spherical tiling we start with a basic shape and then modify it to create an interesting tessellation. Now, however, this basic shape must tile the sphere. Of the many possible spherical tiling schemes [8], in this paper we concentrate on the ones with the highest degree of truly three-dimensional symmetries, the ones derived from the most regular polyhedra - the five Platonic solids.

To create an artistic tessellation based on a particular Platonic solid, we take the face shape of that solid, a triangle, square, or pentagon, and modify its contour as we did in the plane. Now, however, the tiles are projected onto the sphere, and the overall symmetry of the object must be preserved, i.e. the tiles must have polyhedral symmetry.

2.3 Polyhedral Symmetries

The *regular polyhedral group* is a point group. A point group is comprised of a set of points in space corresponding to the polyhedron; the elements of the group are a set of symmetry operations from the following transformations: the identity element (E), n-fold rotations (C_n), inversion[1] (i), improper rotations[2] (S_n), and reflections or mirrors (σ). These symmetry operators leave the overall shape of the polyhedron the same, but can permute the points [2].

Because of the duality relationships among the Platonic solids, there are three polyhedral groups - *tetrahedral*, *octahedral/cuboidal*, and *icosahedral/dodecahedral* [4]. In each of these three groups, we can suppress the mirror symmetries and thus have an "*oriented*" version in addition to the basic *non-oriented*, or *straight*, group. To describe our approach, we will use the simplest example, that of the tetrahedron; the techniques described apply to all the polyhedral symmetry groups.

2.4 Tetrahedral Symmetry

The tetrahedral group, or *straight tetrahedron* group, is the set of points of a tetrahedron with 24 symmetry operators. The first symmetry operator in this group is the identity operation (E) that leaves the tetrahedron un-transformed. There are four 3-fold rotational axes (C_3), one through each tetrahedron vertex and its opposing face centroid (Fig. 2a). There are two possible rotations about these axes of $\pm 120°$, and therefore a total of eight different C_3 rotations. Likewise there are three 2-fold rotation axes (C_2) through the midpoints of opposite edges, each with one possible rotation of 180° (Fig. 2b). These C_2 axes are also used for improper rotations (S_4) of $\pm 90°$. Finally, there are six mirror planes (σ_d), one through each tetrahedron edge and the midpoint of its opposing edge (Fig. 2c). This group of 24 symmetry operators is denoted $\{E, 8C_3, 3C_2, 6S_4, 6\sigma_d\}$ [16].

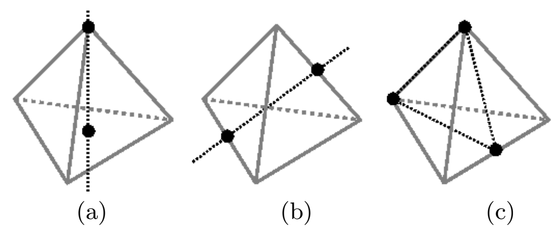

Figure 2: The tetrahedron has (a) four 3-fold axes, (b) three 2-fold axes and improper rotation axes, and (c) six mirror planes.

To see how tetrahedral tiles form, we draw the letter "f" on the triangle faces of the tetrahedron and study how the symmetries cause it to repeat. The face of the tetrahedron has four C_3 points, three C_2 points, and three mirror planes (Fig. 3a), resulting in 6 regions per face for a total of 24 regions. We can think of these regions as tiles of a tessellation. Taking a further look, we see that the mirror symmetries constrain the tile boundary to a fixed location; the tile boundaries cannot be deformed without creating a tessellation that has holes or overlap. So although the tile can be decorated, as with the letter "f", the resulting straight-tile tessellation is not very compelling. However, if the mirror symmetries are eliminated, much more interesting tessellations can be obtained.

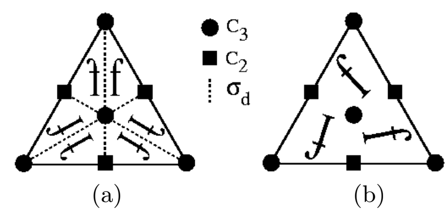

Figure 3: The faces of a tetrahedron with (a) non-oriented symmetry and (b) oriented symmetry.

The *oriented tetrahedron* group has only the rotational symmetries of the tetrahedral group: $\{E, 8C_3, 3C_2\}$. In this new symmetry group, the letter "f" only repeats three times per triangle face, for a total of 12 copies (Fig. 3b). Because the triangular faces no longer contain mirror symmetry constraints, the tile boundaries can now be deformed to create intricate interlocking tile shapes.

The straight and oriented tetrahedron are just two of the seven groups derived from the Platonic solids (Table 1). In addition to the straight and oriented octahedron/cube and icosahedron/dodecahedron, there is also the double tetrahedron group. This group is best described by two interpenetrating tetrahedra of opposite orientation. Of these seven groups, the three oriented ones are best for making interesting organic-looking tile shapes.

[1] Reflection through center of symmetry.

[2] Rotation of 360°/n followed by reflection across the plane perpendicular to the rotation axis.

Table 1: Spherical Symmetry Groups Based on the Platonic Solids

Symmetry Group	Order	Symmetry Operators
`Oriented Tetrahedron`	12	$\{E, 8C_3, 3C_2\}$
`Straight Tetrahedron`	24	$\{E, 8C_3, 3C_2, 6S_4, 6\sigma_d\}$
`Double Tetrahedron`	24	$\{E, 8C_3, 3C_2, i, 8S_6, 3\sigma\}$
`Oriented Octahedron/Cube`	24	$\{E, 8C_3, 6C_2, 6C_4, 3C_4^2\}$
`Straight Octahedron/Cube`	48	$\{E, 8C_3, 6C_2, 6C_4, 3C_4^2, i, 8S_6, 6S_4, 6\sigma_h, 3\sigma_d\}$
`Oriented Icosa/Dodeca-hedron`	60	$\{E, 20C_3, 15C_2, 12C_5, 12C_5^2\}$
`Straight Icosa/Dodeca-hedron`	120	$\{E, 20C_3, 15C_2, 12C_5, 12C_5^2, i, 20S_6, 12S_{10}, 12S_{10}^3, 15\sigma\}$

2.5 Fundamental Tile Domain

Given that the oriented tetrahedron group has twelve distinct symmetry operators, we know that for this class the sphere will be covered with 12 identical tiles. The letter "f" that appeared 12 times across the oriented tetrahedron (Fig. 3b) can be viewed as the placeholder for such a tile. But to what area of the original tetrahedron face does this tile correspond? We can readily split that face into three congruent shapes in many different ways (Fig. 4).

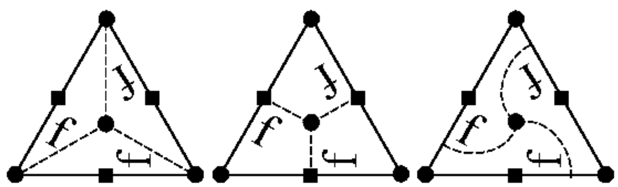

Figure 4: Three possible tilings of the oriented tetrahedron.

At the vertices, where different tiles come together, variants (a) and (b) form quite different patterns. In addition to the vertex of valence 3 formed in the middle of the face, variant (a) also has 3 vertices of valence 2, as well as 3 vertices of valence 6. Variant (b) has 3 vertices of valence 4, as well as 4 vertices of valence 3. Also, in (a) the basic tile shape is triangular, while in (b) it is quadrilateral.

Since the users of our program may be inspired by Escher's work in the plane and may take motifs and tile shapes from these planar patterns, we wanted to make the conversion to the spherical domain as easy as possible. When the preferred tile shape is based on a triangle or a quadrilateral, the user should be able to start with the same basic tile domain on the sphere. We found that it was important to provide both starting patterns, corresponding to variant (a) and (b), even though they belong to the same basic symmetry group. The tiling in Figure 4c can be obtained as a modification of either of the other two schemes.

3. THE PROGRAM

To create an Escher Sphere, the user chooses an initial tile, modifies the tile boundary, and adds detail. On the system side, the program has four major components; a way to help the user select a tiling (Section 3.1); a way to allow the user to modify the tile shape (Section 3.2); a way to decorate the tile with a height field (Section 3.3); and a way to create a solid tile ready for manufacturing (Section 3.4).

3.1 Symmetry Group and Tiling

To begin the user must select one of the predefined symmetry groups. But what does it mean to select the Oriented Octahedron/Cube? Even though cube and octahedron, as well as icosahedron and dodecahedron, are duals of one another and thus are in the same symmetry group, their appearance is quite different. Because it is difficult to understand the meaning of different spherical symmetries, we have the user start by selecting from the more familiar Platonic solids. We find that users often have a basic tile shape in mind, or even a planar Escher tiling, from which they want to start their own explorations. The explicit Platonic shapes provide a more intuitive starting point.

Thus, in our program the seven spherical symmetry groups expanded into eleven starting shapes: the straight and oriented versions of each Platonic solid, plus the special double tetrahedron. Within any one of these shapes, the user must select a particular tile domain. This step has been introduced because we wish to provide users with a basic tile which they then deform. In the case of the oriented tetrahedron tessellated by 12 tiles, we provide the user with the two starting tiles as show in Figure 4a and 4b.

3.2 Modifying the Tile Boundary

After selecting a tile, the user is given a highlighted tile boundary to deform. This tile is drawn on a sphere along with the complete resulting tessellation. The user can deform the tile boundary by inserting points and moving them around on the sphere. The tile boundary is constrained to pass through the vertices of the basic tile, but an arbitrary number of points can be inserted along the edges. As the user manipulates a point, the system automatically adjusts all corresponding points on the whole sphere, based on the chosen symmetry (Fig. 5).

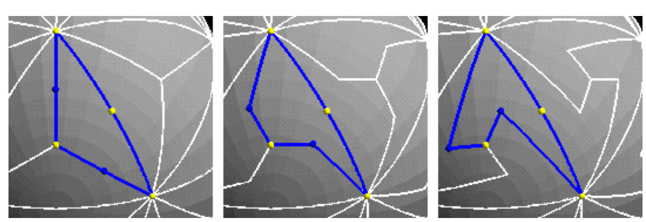

Figure 5: Corresponding tile boundary points move concurrently to maintain symmetry.

Internally, the edges of the basic tile are enumerated and edge correspondence has been precalculated for each group. Thus, when a new point is inserted, the system knows where other points must be inserted to maintain symmetry. All corresponding points have references to each other so that when the user moves a point, all corresponding points are moved concurrently. In this way, the user always sees a complete, flawlessly tiled sphere.

3.3 Adding a Bas-Relief

In addition to changing the tile boundary, points and line segments can also be added for interior detail - for example, to create eyes or fins for a fish tile. Since most SFF processes can only produce uniformly colored parts, distinguishing features within a tile have to be provided as geometry. Following Escher's carving paradigm, it is thus natural to decorate the tiles with a bas-relief. To keep matters simple and reduce fabrication problems, we restrict any such tile modification to adjustments of a radial height field.

Because it is difficult to manipulate three-dimensional points with a two-dimensional input device and display, editing the tile boundary and adjusting the height offsets have been separated into different modes. To create a bas-relief, the user first adds an arbitrary number of interior points to the tile. By direct manipulation, the height of these points can be adjusted individually or in groups (Fig. 6). This makes it easy to add hillocks, ridges, or grooves, and to shape the tile surfaces in a rather naturalistic manner – if so desired.

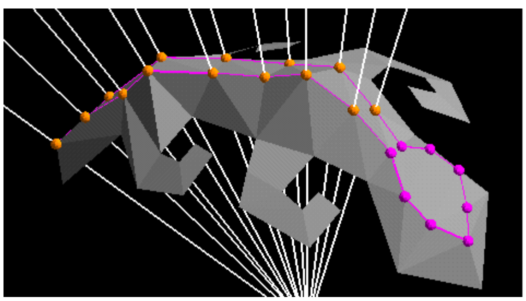

Figure 6: Modifying radial height offsets to create a bas-relief.

Because the points are added individually, the resulting surface is a relatively coarse polyhedral description. An obvious enhancement of the user interface would be to introduce some smooth surface modification schemes in which a region of selectable radius can be pulled or pushed with a weighting factor that falls off radially with the distance from the action point [6][7].

3.4 Creating a Solid Tile

The tile surface is described as a triangle mesh of all the user-defined points, in the interior of the tile and on its boundary. A triangle mesh is created by first stereographically projecting the tile boundary, interior points, and interior lines onto the plane touching the sphere at the tile centroid. These 2D points are then triangulated with Shewchuk's constrained Delaunay triangulation [14] and the calculated connectivity is used to triangulate the 3D points. This scheme has worked well in all our designs. Tiles that reach half-way around the sphere are necessarily quite skinny, and thus the projective distortions do not strongly affect the resulting triangulation. It is thus not necessary to calculate the Delaunay triangulation on the sphere.

Finally, the surface is radially extruded, inward or outward, to create a solid. The bottom surface of the solid can be an offset of the top surface or it can be spherical so that the tiles can be glued onto a round surface.

4. RESULTS

With our system we have successfully designed and fabricated several Escheresque sculptures. The color plate shows pictures of manufactured Escher spheres using fish and lizard tiles.

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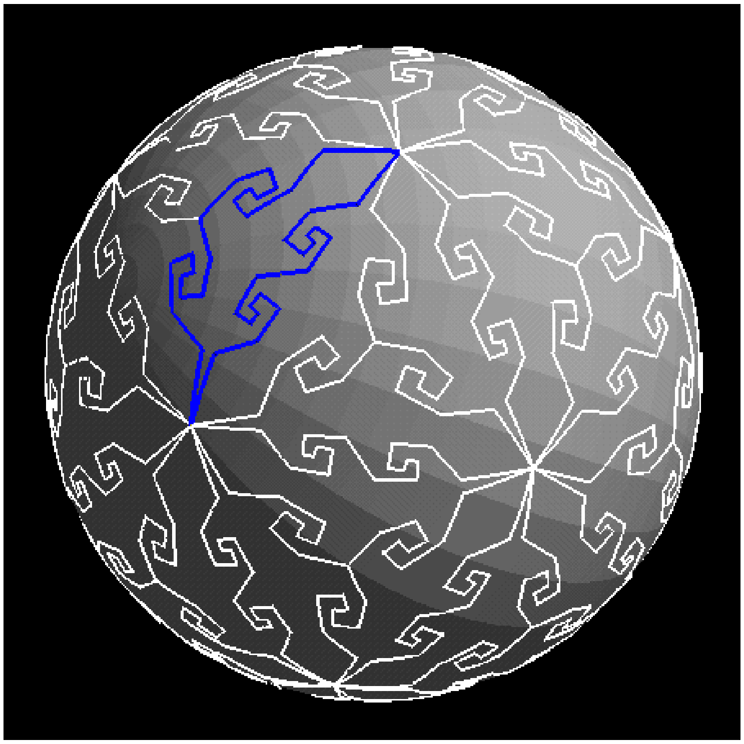

Plate 1. Editing a 60 tile sphere with icosahedral symmetry.

Plate 2. Tiles fresh out of FDM machine.

Plate 3. Tiles freed from their support and joined in interlocking pairs.

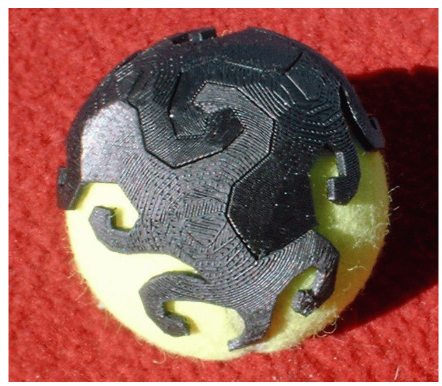

Plate 4. Tiles fit around a tennis ball.

Plate 5. Three different types of Escher spheres with 12 tiles.