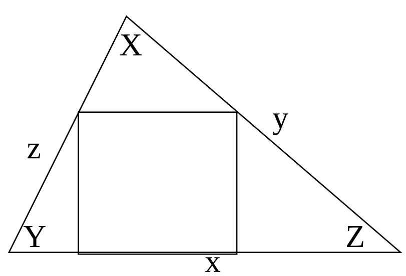
1. Exhibit n and n positive integers $k_1, k_2, \ldots, k_n$ whose sum $k_1 + k_2 + \ldots + k_n = 174$ and whose product $k_1 \cdot k_2 \cdot \ldots \cdot k_n$ is as big as possible, and explain why.

Solution: $n = 174/3 = 58$, and $k_1 = k_2 = \ldots = k_n = 3$. To see why, observe first that every $k_j \ge 2$ since otherwise any $k_j = 1$ could be added to some other and increase the product. Observe second that every $k_j \le 4$ since $k = 2m+1 < (m+1)m$ if integer $m \ge 2$, and $k = 2m < m^2$ if integer $m \ge 3$. Observe third that any $k_j = 4$ can be replaced by two k's equal to 2 without changing the sum nor the product, though n increases by 1. Therefore only 2's and 3's need appear among the k's. Observe fourth that replacing any three k's equal to 2 by two k's equal to 3 increases the product by a factor $9/8$ without changing the sum, though n decreases by 1. Therefore at most two 2's need appear among the k's. But neither $174–2$ nor $174–4$ is divisible by 3, so no 2's appear among the k's; they are all 3.
(E.F. Krause (1996) *Math. Magazine* **69** #4 (Oct.) 270-278)

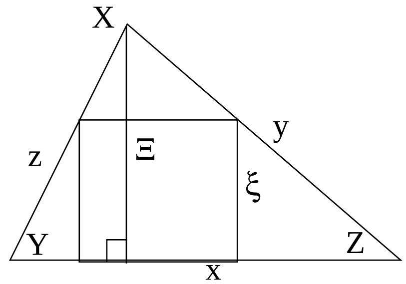
2. Prove that $\sum_{0\le k\le 4n-3} \exp(2\pi\imath k^m/(4n-2)) = 0$ for all positive integers m and n. Here $\imath^2 = -1$, and Euler's formula $\exp(\imath\cdot x) = \cos(x) + \imath\cdot\sin(x)$ may be used.

Proof: After some experiments with small values of m and n it becomes apparent that
$\sum_{0\le k\le 4n-3} \exp(2\pi\imath k^m/(4n-2)) = \sum_{0\le k\le 2n-2} \big(\exp(2\pi\imath k^m/(4n-2)) + \exp(2\pi\imath(k+2n-1)^m/(4n-2)) \big)$
and, as we shall see soon, every term in the latter sum vanishes. To this end observe first that $L := ((k+2n-1)^m - k^m)/(2n-1)$ must be a positive odd integer; L is an integer because of the identity $(K^m - k^m)/(K-k) = K^{m-1} + K^{m-2} k + K^{m-3} k^2 + \ldots + K k^{m-2} + k^{m-1}$. And L is odd because just one of $(k+2n-1)$ and k can be odd. Consequently the asserted result follows from
$\exp(2\pi\imath(k+2n-1)^m/(4n-2)) = \exp\big(2\pi\imath(k^m + (2n-1)L)/(4n-2) \big) = (-1)^L\cdot\exp(2\pi\imath k^m/(4n-2))$.
(J.P.R. Christensen & M.E. Larsen (1996) *Math. Magazine* **69** #4 (Oct.) 305 & 310)

3. Let the side-lengths x, y, z of an acute-angled triangle be so ordered that $x \ge y \ge z > 0$. Of the three inscribed squares each erected on one of the triangle's sides, which is biggest? Why?



Solution: The biggest square is on the smallest side. To see why, let the triangle's angles be X, Y, Z opposite sides with lengths x, y, z respectively, and let ξ be the side-length of the square erected on the side of length x, and let Ξ be the length of the perpendicular dropped from X, as shown here:



From similarity of triangles follows that $(\Xi-\xi)/\xi = \Xi/x$, whence $\xi = x\Xi/(\Xi+x) = 2\Delta/(\Xi+x)$, wherein Δ is the triangle's area. Thus, the biggest square is erected on the side whose length plus perpendicular add up to the least. Let H be the length of the perpendicular dropped from Y onto y, so $\eta = 2\Delta/(H+y)$ is the side-length of the square opposite Y. Then

$$2\Delta/\xi - 2\Delta/\eta = (\Xi+x) - (H+y) = (y\cdot\sin(Z) + x) - (x\cdot\sin(Z) + y) = (x-y)(1-\sin(Z)) > 0 ,$$

so $\eta > \xi$. Similarly the side-length of the square opposite Z is $\zeta > \eta$; it is biggest as claimed.
(E. Sard (1996) *Math. Magazine* **69** #4 (Oct.) 305 & 310)

4. The *Combinatorial Coefficient* is ${}^nC_k := n!/(k!\cdot(n-k)!)$ for integers $n \ge k \ge 0$. Prove that

$$\left(n - k + \sqrt[k]{(k!)} \right)^k/k! \le {}^nC_k \le \left(n - (k-1)/2 \right)^k/k! .$$

Proof: The inequalities to be proved are equalities if $k = 0$ or $k = 1$, so suppose $n \ge k \ge 2$. The proof will invoke the well-known *Arithmetic-Geometric Means Inequality*:

If every $x_j > 0$ then $\sqrt[k]{(x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_k)} \le (x_1+x_2+x_3+\ldots+x_k)/k$.

First set $f(z) := k+z - \sqrt[k]{((z+1)\cdot(z+2)\cdot(z+3)\cdot\ldots\cdot(z+k))}$ for all $z \ge 0$ and let $x_j := 1/(z+j)$ to discover that $f'(z) \le 0$ and consequently $(k-1)/2 = f(\infty) < f(z) \le f(0) = k - \sqrt[k]{(k!)}$. Then set $z := n-k$ to infer that ${}^nC_k\cdot k! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n+1-k) = (n - f(n-k))^k$ lies above $(n - f(0))^k$ and below $(n - f(\infty))^k$, as requested. (When k is big, $f(0) - f(\infty)$ is roughly $1/2 + (1/2 - 1/e)\cdot k$, but that is a story for another day.)

These four problems were presented for solution in two hours. Of course, hardly anybody can do that. However, the Putnam exam is like this; it presents six problems for solution in three morning hours, and six more for solution in three afternoon hours, and hardly anybody can do that. Among about two thousand contestants perhaps half solve no more than one or two out of twelve problems, yet there are usually several contestants who solve most of the problems. If you solved one of these four problems you are about average; if two, superior; if three, wow!