Problem 1: Enumerating Ordered Pairs of Positive Integers

Supply fast arithmetical procedures to *Enumerate* provably all ordered pairs of positive integers. These procedures must achieve a *Bijection* (a 1-to-1 invertible map) between the set of *all* positive integers k and the set of *all* ordered pairs **(**i, j**)** of positive integers thus:

$$k := £(\mathbf{(}i, j\mathbf{)}) \text{ is the } \textit{Label} \text{ of integer pair } \mathbf{(}i, j\mathbf{)} \text{ ; and}$$
$$\mathbf{(}i, j\mathbf{)} := \mathbf{IJ}(k) \text{ is the pair of positive integers labelled by } k .$$

Ideally, the correctness of these procedures will be confirmed by *proofs* that

$$£(\mathbf{IJ}(k)) = k \quad \text{and} \quad \mathbf{IJ}(£(\mathbf{(}i, j\mathbf{)})) = \mathbf{(}i, j\mathbf{)}$$

for all positive integers i, j and k . Moreover each procedure must be “fast” in the sense that the computation time is practically independent of k until it exceeds the biggest integer upon which your computer’s or calculator’s hardware performs arithmetic operations atomically.

Solution 1: Here are two fast simple procedures for all positive integers i, j and k :

$$£(\mathbf{(}i, j\mathbf{)}) := i + (i+j-2)\cdot(i+j-1)/2 \quad \text{maps ordered pair } \mathbf{(}i, j\mathbf{)} \text{ to label } k .$$

$$L(k) := \lfloor 1/2 + \sqrt{2k-1} \rfloor \text{ ;} \qquad (\lfloor x \rfloor \text{ is the biggest integer no bigger than } x)$$
$$M(k) := k - (L(k)-1)\cdot L(k)/2 \text{ ;}$$
$$\mathbf{IJ}(k) := \mathbf{(}\, M(k),\ L(k)-M(k)+1 \,\mathbf{)} \quad \text{maps label } k \text{ to ordered pair } \mathbf{(}i, j\mathbf{)} .$$

Motivation for formula £(**(**i, j**)**) is best revealed by plotting its values at points **(**i, j**)** in the plane, but motivation is not proof. The proof below is based upon properties of *Triangular Numbers* :

$$T_{j+1} := (j+1)\cdot j/2 = 1+2+3+\ldots+(j-1)+j = T_j + j \quad \text{for } j = 0, 1, 2, 3, \ldots .$$

($T_0 = T_1 = 0$.) These numbers partition the set of all positive integers k into disjoint intervals

$$T_j < k \le T_{j+1} \quad \text{for } j = 1, 2, 3, \ldots ,$$

into some one of which every positive integer k must fall. Given k we find $j = L(k)$ satisfies the last two inequalities because L(k) is a monotone nondecreasing function of k that satisfies

$$L(T_j + 1) = j = L(T_{j+1}) \quad \text{for all positive integers } j ,$$

as can be verified by substitution and the employment of elementary inequalities. Do so**!**

The formula for L(k) would still work if $\sqrt{2k-1}$ were replaced by $\sqrt{2k-7/4}$, and its verification would become simpler; but then rounding errors could spoil the formula for very big values k . As it is now, L(k) is easily proved correct *despite roundoff* so long as 2k is less than the smallest positive integer 1000....0001 that the computer’s floating-point arithmetic hardware cannot hold exactly.

Proof: Suppose $k = £(\mathbf{(}i, j\mathbf{)})$; then $T_{i+j-1} < k = £(\mathbf{(}i, j\mathbf{)}) = i + T_{i+j-1} \le T_{i+j}$, so $L(k) = i+j-1$ and then $M(k) = k - T_{L(k)} = i$ and consequently $\mathbf{IJ}(k) = \mathbf{(}i, j\mathbf{)}$ as desired. On the other hand, suppose $\mathbf{(}i, j\mathbf{)} = \mathbf{IJ}(k)$; then $i+j-1 = L(k)$, and consequently $£(\mathbf{(}i, j\mathbf{)}) = M(k) + T_{L(k)} = k$ as desired. Thus the formulas’ correctness is confirmed.

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Another procedure obtains $k-1 = \ldots i_5 j_5 i_4 j_4 i_3 j_3 i_2 j_2 i_1 j_1 i_0 j_0$ by interleaving the digits of $i-1 = \ldots i_5 i_4 i_3 i_2 i_1 i_0$ and $j-1 = \ldots j_5 j_4 j_3 j_2 j_1 j_0$, and conversely. Sub-procedures for extracting digits and reassembling them have to be proved correct; these are tedious to describe and slower on almost all modern computers than the procedures above.

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Problem 2: Enumerating Positive Rational Numbers
The positive rational numbers $r = m/n$ could be identified just with the pairs (m, n) of positive integers were it not necessary to reduce m and n to "lowest terms" by cancelling out their common factors in order to represent every rational r uniquely. For that reason, enumerating pairs (m, n) proves that the rationals are countable but does not provide an enumeration of them.

Provide an explicit enumeration in the form of a pair of functions £(r) and R(k) defined for *all* rationals $r > 0$ and integer indices $k > 0$, computable in a time short compared with the integer label k = £(r) when it grows huge, and inverse in the sense that r = R(£(r)) and k = £(R(k)) .

Can you see why interlacing the digits of a rational number's numerator and denominator (even if first decremented by 1) into one integer does not meet our requirements? Hint: 222222.

Solution 2: To obtain an explicit enumeration of the positive rationals r , we express $1 + 1/r$ as a *Terminating Continued Fraction* $1 + 1/r = a + 1/(b + 1/(c + 1/(\dots i + 1/(j+1) \dots)))$ in which each of a, b, c, ..., i and j is a positive integer determined by a well-known repetitive process:

$$a := \lfloor 1 + 1/r \rfloor ;$$
$$b := \lfloor 1/(1 + 1/r - a) \rfloor ;$$
$$c := \lfloor 1/(1/(1 + 1/r - a) - b) \rfloor ;$$
$$\dots .$$

Here the rational numbers of which integer parts are taken have numerators and denominators that shrink in the course of the process, so it must terminate; look up *Euclid's GCD Algorithm* in textbooks or `<www.cs.berkeley.edu/~wkahan/MathH110/gcd5.pdf>`. The last integer divisor $j+1$ exceeds 1 for the sake of the continued fraction's uniqueness.

Thus, every positive rational r can be associated with a finite sequence $(a, b, c, ..., i, j)$ of positive integers, and *vice-versa*; and the association is *bijective* because different sequences go with different rationals. Next we associate every such finite sequence of positive integers with a finite strictly increasing sequence of nonnegative integers $(A, B, C, ..., I, J)$ thus:

$$A := a - 1 ; \quad B := A + b ; \quad C := B + c ; \quad \dots ; \quad J := I + j .$$

This association is bijective too because it is reversible:

$$j = J - I ; \quad \dots; \quad c = C - B ; \quad b = B - A ; \quad a = A + 1 .$$

Therefore a bijection has been constructed between the positive rationals r and the finite strictly increasing sequences $(A, B, C, ..., J)$ of nonnegative integers. Now associate these sequences bijectively with the binary expansions of positive integer indices

$$k := 2^A + 2^B + 2^C + \dots + 2^I + 2^J .$$

Thus, a way has been exhibited to compute quickly a positive integer label k = £(r) for every positive rational r , and inversely to compute quickly a positive rational r = R(k) for every positive integer k . Evidently £(R(k)) = k and R(£(r)) = r for all rationals $r > 0$ and integers $k > 0$, so this is an explicit enumeration of the kind desired. The time taken to compute those functions is roughly proportional to the number of nonzero bits in the binary expansion of k , which grows slowly (logarithmically) with k as it tends to infinity.

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A simpler alternative, at first sight, is to compute k = £(r) := $2^{a-1} \cdot 3^{b-1} \cdot 5^{c-1} \cdot \dots$ as a finite product of prime powers. However, to compute R(k) then we would have to factor k ; but currently nobody knows how to factor gargantuan integers k faster than in a time proportional to at least $\sqrt[3]{k}$ instead of log(k) .

Problem 3: Linearizing Ordered Pairs of Real Numbers
"Linearizing" is better than "Enumerating". What is desired here is an explicit bijection between two uncountable sets: One is the set $\mathbb{R}^+$ of all positive real numbers r ; the other is its set $\mathbb{R}^{+2}$ of all ordered pairs **(**s, t**)** of positive reals. ("Ordered" distinguishes **(**s, t**)** from **(**t, s**)** .) In other words, the problem is to construct a bijection between the positive ray on the real axis and the positive orthant in the plane.

Solution 3: Construct a bijection by interlacing digits. Let decimal expansions of r, s and t be $r = \dots R_3R_2R_1R_0.r_1r_2r_3\dots$, $s = \dots S_3S_2S_1S_0.s_1s_2s_3\dots$ and $t = \dots T_3T_2T_1T_0.t_1t_2t_3\dots$ respectively. Setting $\dots R_3R_2R_1R_0.r_1r_2r_3\dots := \dots S_3T_3S_2T_2S_1T_1S_0T_0.s_1t_1s_2t_2s_3t_3\dots$ is too easy; it fails to achieve the desired bijection for two reasons both illustrated by r = 12/11 = 1.090909… , which maps to **(**s, t**)** = **(**0, 1.999…**)** = **(**0, 2**)** , neither of them positive pairs, while duplicating the map from t = 2.0 . To correct these failures let us institute two measures:

First, we prevent infinite tails of zeros by choosing for each positive terminating decimal number its alternate representation with an infinite tail of nines; for example, for 2.6 = 2.5999… we shall choose the latter decimal expansion. Second, instead of interleaving decimal digits after the decimal point, we shall interleave *Digit-Blocks* consisting of some number (perhaps none) of consecutive zeros followed by a nonzero digit. For example, 120.30400500067080999… breaks into |1|2|0|**.**|3|04|005|0006|7|08|09|9|9|… when broken into digit-blocks. Now let the expansion of r into digit-blocks be $r = \dots R_3R_2R_1R_0.r_1r_2r_3\dots$, and similarly for s and t . Then setting $\dots R_3R_2R_1R_0.r_1r_2r_3\dots := \dots S_3T_3S_2T_2S_1T_1S_0T_0.s_1t_1s_2t_2s_3t_3\dots$ achieves the desired explicit (and surprisingly simple) bijection.

Problem: Use the foregoing bijection between $\mathbb{R}^+$ and $\mathbb{R}^{+2}$ to construct an explicit bijection between the set $\mathbb{R}$ of all real numbers (positive, negative and zero) and its set $\mathbb{R}^2$ of all pairs of real numbers. Hint: ln and exp .

Problem: What about a bijection between the set $\mathbb{R}^o$ of all nonnegative real numbers and its set $\mathbb{R}^{o2}$ of pairs? Hint: Zero is an integer.

Problem: What about a bijection between the set $\mathbb{S}^3$ of all ordered triples from a set $\mathbb{S}$ given a bijection {£, **IJ**} between $\mathbb{S}$ and its set $\mathbb{S}^2$ of ordered pairs?

Actually, a bijection *exists* between any infinite set $\mathbb{S}$ and its set $\mathbb{S}^2$ of ordered pairs, but there may be no way to *construct* the bijection because the proof of its existence depends upon the *Axiom of Choice*. We have no neat proof.

Problem 4: Rearranging the Order of Summation of a Doubly-Summed Infinite Series
Suppose every $x_{m,n} \ge 0$. Show why, if either $\sum_{m\ge1}\sum_{n\ge1} x_{m,n}$ or $\sum_{n\ge1}\sum_{m\ge1} x_{m,n}$ converges, both converge to the same sum.

(Like a singly-summed series, a doubly-summed series $\sum_{m\ge1}\sum_{n\ge1} q_{m,n}$ is said to converge *Absolutely* if $\sum_{m\ge1}\sum_{n\ge1} |q_{m,n}|$ converges too. Without absolute convergence, rearranging the order of summation can change the sum of a doubly-summed series. For example let $q_{k,n} := 0$ unless $|k-n| = 1$ and then $q_{k,n} := k-n$; now $\sum_{k\ge1}\sum_{n\ge1} q_{k,n} = -1 \ne 1 = \sum_{n\ge1}\sum_{k\ge1} q_{k,n}$.)

Proof 4: The proof converts each given doubly-summed series to a singly-summed series by using any *Enumeration* {£(i, j), **IJ**(k)} of pairs; this is a pair of functions that implement a bijection between the set of all positive integers k and the set of all ordered pairs (i, j) of them: Positive integer k = £(i, j) if and only if positive integer pair (i, j) = **IJ**(k) .

Use the enumeration to set $X_{£(i,\,j)} := x_{i,j}$. Our task is to prove that …

$$s := \sum_{m\ge1} \sum_{n\ge1} x_{m,n} \ \text{ converges if and only if } \ S := \sum_{k\ge1} X_k \ \text{ converges, and then } \ s = S .$$

First suppose s converges. Let $S_K := \sum_{1\le k\le K} X_k$ and, for any chosen large integer K , choose M and N big enough that $M \ge m$ and $N \ge n$ for every pair (m, n) = **IJ**(k) with $1 \le k \le K$. Then $S_K = \sum_{1\le k\le K} X_k \le \sum_{1\le m\le M} \sum_{1\le n\ N} x_{m,n} \le \sum_{m\ge1} \sum_{n\ge1} x_{k,n} = s$ regardless of K , whence follows that S converges and $S \le s$.

On the other hand suppose S converges. Next, each $s_m := \sum_{n\ge1} x_{m,n}$ will be proved convergent as follows: For any chosen integers $m \ge 1$ and large N choose K so big that all pairs (m, n) with $1 \le n \le N$ appear among the pairs (m, n) = **IJ**(k) when $1 \le k \le K$. Then, regardless of N , the sum $\sum_{1\le n\le N} x_{m,n} \le \sum_{1\le k\le K} X_k \le S$, so s_m converges. Finally, for any chosen large integers M and N choose K so big that all pairs (m, n) with $1 \le m \le M$ and $1 \le n \le N$ appear among the pairs (m, n) = **IJ**(k) for $1 \le k \le K$. Now $\sum_{1\le m\le M} \sum_{1\le n\ N} x_{m,n} \le \sum_{1\le k\le K} X_k \le S$ regardless of M and N . Let $N \to \infty$ to deduce that $\sum_{1\le m\le M} s_m \le S$ regardless of M , whence follows that $s = \sum_{m\ge1} s_m$ converges and $s \le S$. Therefore $s = S$ if either of s and S converges.

The process that converted $s := \sum_{m\ge1} \sum_{n\ge1} x_{m,n}$ to $S := \sum_{k\ge1} X_k$ also converts $\sum_{n\ge1} \sum_{m\ge1} x_{m,n}$ to S ; so all three sums converge and are equal if any one of them converges.

Swapping absolutely convergent doubly-summed series should be covered in Math. 104 but usually isn't.

Problem 5: Suppose that every $c_n \ge 0$ and that $ç := \sum_{n\ge1} c_n/n$ converges. Explain why $Ç := \sum_{k\ge1} \sum_{n\ge1} c_n/(n^2 + k^2)$ must converge too.

Solution 5: Swapping the order of summation in the doubly-summed series Ç has been justified in Problem 4. Swapping now produces the doubly-summed series

$$G := \sum_{n\ge1} \sum_{k\ge1} c_n/(n^2 + k^2) = \sum_{n\ge1} c_n \cdot \sum_{k\ge1} 1/(n^2 + k^2) = \sum_{n\ge1} c_n \cdot g_n ,$$

where $g_n := \sum_{k\ge1} 1/(n^2 + k^2)$. Because $1/(n^2 + k^2)$ decreases as k increases, each

$$g_n < \int_0^\infty dk/(n^2 + k^2) = (\arctan(\infty) - \arctan(0))/n = \pi/(2n) . \qquad \text{(Do you see why?)}$$

Consequently $G < ç\cdot\pi/2$; in short, G is an *Absolutely Convergent* doubly-summed series. Now G 's absolute convergence implies that Ç converges to Ç = G .

Problem 6: Assume positive real numbers $x_1, x_2, \ldots, x_n$ are so big that $\sum_{1\le k\le n} 1/(1+x_k^2) = 1$, so $n \ge 2$. Prove $\sum_{1\le k\le n} x_k \ge (n-1)\cdot\sum_{1\le k\le n} 1/x_k$ with equality only if $n = 2$ or each $x_k = \sqrt{n-1}$.
Hint: $\sum_j 1/(1+x_j^2)\cdot\sum_k x_k - \sum_j x_j^2/(1+x_j^2)\cdot\sum_k 1/x_k$

Proof 6: Since the problem's assumption implies that $\sum_k x_k^2/(1+x_k^2) = n-1$, we can write

$$\sum_k x_k - (n-1)\cdot\sum_k 1/x_k = \sum_j 1/(1+x_j^2)\cdot\sum_k x_k - \sum_j x_j^2/(1+x_j^2)\cdot\sum_k 1/x_k =$$

$$= \sum_j \cdot\sum_k (x_k - x_j^2/x_k)/(1+x_j^2) = \sum_j \cdot\sum_k (x_k^2 - x_j^2)/\big((1+x_j^2)\cdot x_k\big) =$$

Now swap j with k and average the sums.

$$= \tfrac{1}{2}\cdot\sum_j \cdot\sum_k (x_k^2 - x_j^2)\cdot\big(\, 1/\big((1+x_j^2)\cdot x_k\big) - 1/\big((1+x_k^2)\cdot x_j\big)\,\big) =$$

$$= \tfrac{1}{2}\cdot\sum_j \cdot\sum_k (x_k^2 - x_j^2)\cdot(x_k - x_j)\cdot(x_k\cdot x_j - 1)/\big(\,(1+x_j^2)\cdot(1+x_k^2)\cdot x_j\cdot x_k\,\big) =$$

$$= \tfrac{1}{2}\cdot\sum_j \cdot\sum_k (x_k + x_j)\cdot(x_k - x_j)^2\cdot(x_k\cdot x_j - 1)/\big(\,(1+x_j^2)\cdot(1+x_k^2)\cdot x_j\cdot x_k\,\big)\,.$$

Because every $1/(1+x_j^2) + 1/(1+x_k^2) \le 1$ if $j \ne k$, so is $2 + x_k^2 + x_j^2 \le 1 + x_k^2 + x_j^2 + x_k^2\cdot x_j^2$, whence follows that every such $x_k\cdot x_j \ge 1$. Therefore $\sum_k x_k - (n-1)\cdot\sum_k 1/x_k \ge 0$ with equality only if $n = 2$ (and then $x_2\cdot x_1 = 1$) or every $x_k = x_j = \sqrt{n-1}$, as claimed.

This problem is an exercise in double summation posed by W. Janous in 1991 and solved on pp. 678-9 of *The Amer. Math. Monthly* **99** #7 (Aug.-Sept. 1992).