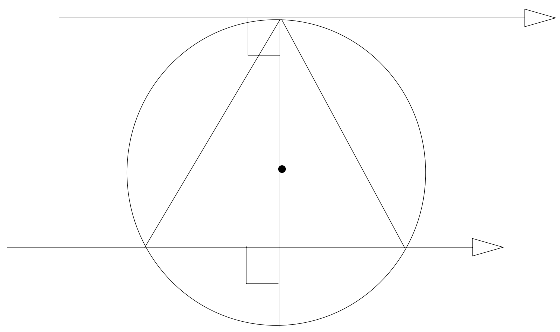
Problem 1: Given is an ellipse E neither a circle nor degenerate (*i.e.* a straight line segment). Let Δ be the largest of the areas of triangles inscribed in E . How many inscribed triangles have maximal area Δ ? At least two do since E is centrally symmetric. Are there more? Why?

Solution 1: There are infinitely many triangles of maximal area Δ inscribed in E , every boundary-point of which is a vertex of one such triangle. Here is why:

$E = L^{-1}\cdot O$ is the image of a circle O mapped by some invertible linear operator L^{-1}. This means that, after we move E and O to center both of them at the origin $\mathbf{o}$, running $\mathbf{x}$ through all 2-vectors of Euclidean length $||\mathbf{x}|| = \rho$, the positive radius of O , runs $L^{-1}\cdot\mathbf{x}$ around E as $\mathbf{x}$ runs around O . We may choose ρ to make O have the same area as E has, and then L can be chosen to have $\det(L) = 1$ so that area is preserved by operators L and L^{-1} . Any triangle T of maximal area Δ inscribed in E is the image of $L\cdot T$, a triangle of maximal area Δ inscribed in O . Which inscribed triangles have maximal area? This question's answer is almost obvious:



At each vertex of a triangle of maximal area inscribed in O the tangent to O must be parallel to the triangle's opposite side; otherwise the vertex could be moved slightly, without moving the opposite side, to increase the triangle's altitude and therefore its area. Therefore a perpendicular dropped from a vertex to the triangle's opposite side must pass through the circle's center, which makes the circle and the triangle each its own reflection in that perpendicular. Consequently any two sides of the triangle must have equal lengths. We conclude that every triangle of maximal area Δ inscribed in O is equilateral; every point of O is a vertex of one of them. And L^{-1} maps every one of them onto a triangle of maximal area Δ inscribed in E , as claimed.

This solution takes far longer to read than to visualize after you have seen it.
Infinitely many tetrahedra of maximal volume are inscribed in an ellipsoid for similar reasons.

The next two problems have lengthy solutions which, if you cannot find them for dimension n in general, should be solved first for dimension n = 2 , then n = 3 , in order to get the idea.

Problem 2: In a Euclidean n-space, where a vector $\mathbf{v}$ has length $\|\mathbf{v}\| := \sqrt{(\mathbf{v}^T\cdot\mathbf{v})}$, a reflection in a line ($n = 2$) or (hyper)plane ($n \ge 3$) through the origin $\mathbf{o}$ must be a linear operator of the form $V := \mathbf{I} - 2\cdot\mathbf{v}\cdot\mathbf{v}^T/\mathbf{v}^T\cdot\mathbf{v} = V^T = V^{-1}$ wherein $\mathbf{v}$ is any vector perpendicular to the mirror. (Can you see why? Think about $V\cdot\mathbf{v}$, and about $V\cdot\mathbf{x}$ when $\mathbf{v}^T\cdot\mathbf{x} = 0$.) The product $R := V\cdot W$ of any two such reflections is a rotation; $R^T = R^{-1}$ and $\det(R) = 1$. Given four nonzero vectors $\mathbf{x}$, $\mathbf{y} \ne \mathbf{x}$, $\mathbf{s}$ and $\mathbf{t} \ne \mathbf{s}$ with $\|\mathbf{x}\| = \|\mathbf{y}\|$, $\|\mathbf{s}\| = \|\mathbf{t}\|$ and $\mathbf{s}^T\cdot\mathbf{x} = \mathbf{t}^T\cdot\mathbf{y}$, so that $|\angle(\mathbf{s}, \mathbf{x})| = |\angle(\mathbf{t}, \mathbf{y})|$, show how and why to construct reflections V and W so that $R := V\cdot W$ will rotate $\mathbf{x}$ to $R\cdot\mathbf{x} = \mathbf{y}$ and $\mathbf{s}$ to $R\cdot\mathbf{s} = \mathbf{t}$, provided such an R exists. When does it not exist? When is R unique?

Solution 2: A requested $R := V\cdot W$ can be a product of two reflections $W := \mathbf{I} - 2\cdot\mathbf{w}\cdot\mathbf{w}^T/\|\mathbf{w}\|^2$ and $V := \mathbf{I} - 2\cdot\mathbf{v}\cdot\mathbf{v}^T/\|\mathbf{v}\|^2$ in which $\mathbf{w} := \mathbf{x} - \mathbf{y}$ and $\mathbf{v} := W\cdot\mathbf{s} - \mathbf{t}$, except that if $W\cdot\mathbf{s} = \mathbf{t}$ then $\mathbf{v}$ may be any nonzero vector orthogonal to both $\mathbf{y}$ and $\mathbf{t}$ provided such a vector exists. Here is why this works when it works: First confirm that W swaps $\mathbf{x}$ and $\mathbf{y}$ by introducing $\mathbf{z} := \mathbf{x} + \mathbf{y}$, observing that $\mathbf{w}^T\cdot\mathbf{z} = 0$, and then substituting $\mathbf{x} = (\mathbf{z} + \mathbf{w})/2$ and $\mathbf{y} = (\mathbf{z} - \mathbf{w})/2$ into the two equations $W\cdot\mathbf{x} = \mathbf{y}$ and $W\cdot\mathbf{y} = \mathbf{x}$ to confirm that both are satisfied. Similarly, if $\mathbf{v} = W\cdot\mathbf{s} - \mathbf{t} \ne \mathbf{o}$ then V swaps $W\cdot\mathbf{s}$ and $\mathbf{t}$ while preserving $\mathbf{y}$ because $\mathbf{v}^T\cdot\mathbf{y} = 0$; grind through the algebra. On the other hand, if $W\cdot\mathbf{s} = \mathbf{t}$ and $\mathbf{v} \ne \mathbf{o}$ satisfies $\mathbf{v}^T\cdot\mathbf{y} = \mathbf{v}^T\cdot\mathbf{t} = 0$ then V preserves both $\mathbf{t}$ and $\mathbf{y}$. Either way, $R := V\cdot W$ rotates $[\mathbf{x}, \mathbf{s}]$ to $R\cdot[\mathbf{x}, \mathbf{s}] = V\cdot W\cdot[\mathbf{x}, \mathbf{s}] = V\cdot[\mathbf{y}, W\cdot\mathbf{s}] = [\mathbf{y}, \mathbf{t}]$ as required.

Provided R exists. When can no such R exist? Just when the dimension $n = 2$, and reflection W swaps $\mathbf{s}$ and $\mathbf{t}$ as well as $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{t}$ and $\mathbf{y}$ are linearly independent (neither parallel no antiparallel), in which case $\mathbf{s}$ and $\mathbf{x}$ are linearly independent too because of the constraints $[\mathbf{x}, \mathbf{s}]^T\cdot[\mathbf{x}, \mathbf{s}] = [\mathbf{y}, \mathbf{t}]^T\cdot[\mathbf{y}, \mathbf{t}]$ that were given: These constraints imply for every 2-vector $\mathbf{b}$ that $\|[\mathbf{x}, \mathbf{s}]\cdot\mathbf{b}\|^2 = \|[\mathbf{y}, \mathbf{t}]\cdot\mathbf{b}\|^2$, so neither $[\mathbf{x}, \mathbf{s}]\cdot\mathbf{b}$ nor $[\mathbf{y}, \mathbf{t}]\cdot\mathbf{b}$ could vanish unless the other did too. Then $W\cdot\mathbf{s} = \mathbf{t}$ but no $\mathbf{v} \ne \mathbf{o}$ can satisfy $\mathbf{v}^T\cdot[\mathbf{y}, \mathbf{t}] = [0, 0]$; instead the nonsingular linear equation $R\cdot[\mathbf{x}, \mathbf{s}] = [\mathbf{y}, \mathbf{t}] = W\cdot[\mathbf{x}, \mathbf{s}]$ would pre-emptively force $R = W$, a reflection, not a rotation.

What's the difference between Reflection and Rotation? Reflection V has $\det(V) = -1$. This can be proved either by choosing a new orthonormal coordinate system with $\mathbf{v}$ as one of its basis vectors, in which case V becomes a diagonal matrix obtained from the identity matrix $\mathbf{I}$ by reversing the sign of one of its diagonal elements, or else it can be deduced from an important determinantal identity

$$\det(\mathbf{I} - \mathbf{p}\cdot\mathbf{q}^T) = \det\left(\begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{q}^T & 1 \end{bmatrix}\right) = 1 - \mathbf{q}^T\cdot\mathbf{p},$$

left for the diligent student to confirm. And then $\det(R) = \det(V\cdot W) = \det(V)\cdot\det(W) = +1$. In fact every *Proper* rotation R that preserves the left- or right-handed orientation of a basis must have determinant $\det(R) = +1$.

When is R determined uniquely by the given four nonzero vectors $\mathbf{x}$, $\mathbf{y} \ne \mathbf{x}$, $\mathbf{s}$ and $\mathbf{t} \ne \mathbf{s}$ with $\|\mathbf{x}\| = \|\mathbf{y}\|$, $\|\mathbf{s}\| = \|\mathbf{t}\|$ and $\mathbf{s}^T\cdot\mathbf{x} = \mathbf{t}^T\cdot\mathbf{y}$? Not when dimension $n > 3$; simple examples show why. When dimension $n = 2$ and R exists, the data determine it uniquely. This follows easily when $\mathbf{s}$ and $\mathbf{x}$ are linearly independent; then $R\cdot[\mathbf{x}, \mathbf{s}] = [\mathbf{y}, \mathbf{t}]$ implies $R = [\mathbf{y}, \mathbf{t}]\cdot[\mathbf{x}, \mathbf{s}]^{-1}$. Otherwise $\mathbf{s} = ß\cdot\mathbf{x}$ for some scalar $ß \ne 0$, and then putting $\mathbf{b} := [-ß, 1]^T$ above implies $\mathbf{t} = ß\cdot\mathbf{y}$ too, in which case R is determined uniquely and explicitly by $\mathbf{x}$ and $\mathbf{y}$ alone as follows:

Let $\mathbf{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\mathbf{J}^T = -\mathbf{J}^{-1}$; it rotates the plane a quarter turn because $\mathbf{v}^T\cdot\mathbf{J}\cdot\mathbf{v} = 0$ for every 2-

vector $\mathbf{v}$. Just as R rotates $\mathbf{x}$ to $R\cdot\mathbf{x} = \mathbf{y}$ so does it rotate $\mathbf{J}\cdot\mathbf{x}$ to $R\cdot\mathbf{J}\cdot\mathbf{x} = \mathbf{J}\cdot\mathbf{y}$; in other words, plane rotations commute. Consequently $R\cdot[\mathbf{x}, \mathbf{J}\cdot\mathbf{x}] = [\mathbf{y}, \mathbf{J}\cdot\mathbf{y}]$, whence $R = [\mathbf{y}, \mathbf{J}\cdot\mathbf{y}]\cdot[\mathbf{x}, \mathbf{J}\cdot\mathbf{x}]^{-1}$ is determined uniquely by $\mathbf{x}$ and $\mathbf{y}$. An explicit formula $R := \mathbf{I}\cdot\mathbf{x}^T\cdot\mathbf{y}/\mathbf{x}^T\cdot\mathbf{x} - \mathbf{J}\cdot\mathbf{x}^T\cdot\mathbf{J}\cdot\mathbf{y}/\mathbf{x}^T\cdot\mathbf{x}$ is left for the diligent student to confirm. (This R rotates $\mathbf{s}$ to $R\cdot\mathbf{s} = \mathbf{t}$ too even if $\mathbf{s}$ and $\mathbf{x}$ are independent provided such an R exists.)

When dimension $n = 3$ the given data $\mathbf{x}$, $\mathbf{y}$, $\mathbf{s}$ and $\mathbf{t}$ determine R uniquely only if $\mathbf{s}$ and $\mathbf{x}$ are linearly independent. Then R rotates $\mathbf{x}$ to $R\cdot\mathbf{x} = \mathbf{y}$, $\mathbf{s}$ to $R\cdot\mathbf{s} = \mathbf{t}$, and therefore the nonzero cross-product $\mathbf{x}\times\mathbf{s}$ to $R\cdot(\mathbf{x}\times\mathbf{s}) = (R\cdot\mathbf{x})\times(R\cdot\mathbf{s}) = \mathbf{y}\times\mathbf{t}$, whereupon $R = [\mathbf{y}, \mathbf{t}, \mathbf{y}\times\mathbf{t}]\cdot[\mathbf{x}, \mathbf{s}, \mathbf{x}\times\mathbf{s}]^{-1}$.

Problem 3: In a Euclidean n-space, where a vector $\mathbf{v}$ has length $\|\mathbf{v}\| := \sqrt{(\mathbf{v}^T\cdot\mathbf{v})}$, a *Box* is a *Rectangular Parallelepiped*, a figure bounded by 2n flat facets each of which intersects 2n–2 perpendicular facets; none need be parallel to coordinate (hyper)planes ($n \ge 3$) or lines ($n = 2$). The *Diameter* of a box is the distance between any two opposite vertices. An *Ellipse* ($n = 2$) or *Ellipsoid* ($n \ge 3$) centered at $\mathbf{o}$ is the locus of points $\mathbf{x}$ that satisfy an equation of the form $\mathbf{x}^T\cdot H^{-1}\cdot\mathbf{x} = 1$ for some *Symmetric Positive-Definite* matrix H; “symmetric” means $H = H^T$, and “positive definite” means $\mathbf{v}^T\cdot H\cdot\mathbf{v} > 0$ for every n-vector $\mathbf{v} \ne \mathbf{o}$. For any such H, every box that circumscribes the ellipsoid tightly enough for all facets to touch it has the same diameter $2\cdot\sqrt{(\mathrm{Trace}(H))}$ where $\mathrm{Trace}(H) = \sum_j h_{jj}$ is the sum of the diagonal elements of H. Explain why.

Solution 3: Let ❑ denote the n-dimensional unit (hyper)cube ($n \ge 3$) or square ($n = 2$), the convex hull of 2^n vertices each of whose n coordinates are all selected from the set {1. –1} . Every box $\mathbb{B}$ centered at the origin $\mathbf{o}$ is obtained from ❑ by a *Dilatation* and a *Rotation*; this means $\mathbb{B} = R\cdot V\cdot$❑ where n-by-n diagonal matrix V has n positive elements and represents the dilatation, and R is an n-by-n *Proper Orthogonal* matrix ($R^T = R^{-1}$ and $\det(R) = +1$) that represents the rotation. Let $\mathbf{u}$ be the column n-vector whose every element is 1. Just as point $\mathbf{c}$ lies in ❑ just when $|\mathbf{c}| \le \mathbf{u}$ *elementwise* (which means that no element of $\mathbf{c}$ exceeds 1 in magnitude), so does point $\mathbf{b} = R\cdot V\cdot\mathbf{c}$ lie in box $\mathbb{B}$ just when $|V^{-1}\cdot R^{-1}\cdot\mathbf{b}| \le \mathbf{u}$ elementwise.

Changing coordinates to a new orthonormal basis consisting of the columns of R is tantamount to rotating everything in the space by R^{-1}. Whatever point was represented by $\mathbf{x}$ in the original coordinate system is represented by $\mathbf{y} := R^{-1}\cdot\mathbf{x}$ in the new coordinates. Let ❒ denote the new coordinates' unit (hyper)cube; don't confuse it with the old coordinates' ❑. Now the change of coordinates can be construed either as a rotation of box $\mathbb{B} = R\cdot V\cdot$❑ to $R^{-1}\cdot\mathbb{B} = V\cdot$❑, or else as providing a new representation for $\mathbb{B} = V\cdot$❒ in the new coordinates.

A symmetric positive definite matrix H represents an ellipsoid $\mathbb{H}$ as the locus of points $\mathbf{x}$ satisfying $\mathbf{x}^T\cdot H^{-1}\cdot\mathbf{x} = 1$. How does the change to new coordinates alter the representation of $\mathbb{H}$? Substitute $\mathbf{x} = R\cdot\mathbf{y}$ into the equation to get $\mathbf{y}^T\cdot W^{-1}\cdot\mathbf{y} = 1$ for the symmetric positive definite matrix $W := R^{-1}\cdot H\cdot R$ that represents $\mathbb{H}$ in the new coordinates or $R^{-1}\cdot\mathbb{H}$ in the old coordinates. Note that $\mathrm{Trace}(W) = \mathrm{Trace}(H)$. This will be needed later and can be proved by rearranging the ordering of a triple summation; can you do it?

In the new coordinates, $\mathbb{B} = V\cdot❒ \supseteq \mathbb{H}$ just when $V\cdot\mathbf{u} \ge |\mathbf{y}|$ whenever $\mathbf{y}^T\cdot W^{-1}\cdot\mathbf{y} = 1$, and we seek the smallest positive diagonal V for which $\mathbb{B} \supseteq \mathbb{H}$ so that every facet of $\mathbb{B}$ will touch $\mathbb{H}$. This sought V is characterized by the equation $\mathbf{f}^T\cdot V\cdot\mathbf{u} = \{ \max \mathbf{f}^T\cdot\mathbf{y}$ subject to $\mathbf{y}^T\cdot W^{-1}\cdot\mathbf{y} = 1 \}$ for every n-row $\mathbf{f}^T = [0, 0, \dots, 0, 1, 0, \dots, 0, 0]$ whose elements are all zeros but one and it is 1. The desired maxima can be found by using *Lagrange Multipliers*, or more directly as follows:

Because W is positive definite, $(\mathbf{f}\cdot\mu - W^{-1}\cdot\mathbf{y})^T\cdot W\cdot(\mathbf{f}\cdot\mu - W^{-1}\cdot\mathbf{y}) \ge 0$ for every real scalar μ. The inequality's left-hand side expands to a quadratic polynomial in μ that cannot reverse sign, so its diacriminant cannot be positive: $(\mathbf{f}^T\cdot\mathbf{y})^2 \le (\mathbf{f}^T\cdot W\cdot\mathbf{f})\cdot(\mathbf{y}^T\cdot W^{-1}\cdot\mathbf{y})$, with equality achieved when $\mathbf{y} = W\cdot\mathbf{f}/\sqrt{(\mathbf{f}^T\cdot W\cdot\mathbf{f})}$. Consequently $\{ \max \mathbf{f}^T\cdot\mathbf{y}$ subject to $\mathbf{y}^T\cdot W^{-1}\cdot\mathbf{y} = 1 \} = \sqrt{(\mathbf{f}^T\cdot W\cdot\mathbf{f})}$.

Let $\mathbf{f}^T$ have its sole nonzero element in the j^{th} position to see that the sought smallest positive diagonal V has diagonal elements $v_{jj} = \sqrt{w_{jj}}$. All the vertices of $\mathbb{B} = V\cdot❒$ have new coordinates $[\pm v_{11}, \pm v_{22}, \pm v_{33}, \dots, \pm v_{nn}]$ whence follows the desired conclusion that

$$\text{diameter}(\mathbb{B}) = 2\cdot\sqrt{(\textstyle\sum_j v_{jj}^2)} = 2\cdot\sqrt{(\text{Trace}(W))} = 2\cdot\sqrt{(\text{Trace}(H))}\ .$$

Problem 4a: A polynomial $M(z) := \sum_{0\le k\le n} \mu_k\cdot z^k$ whose coefficients μ_k are all integers is called "Irreducible" if it is not the product of two nonconstant polynomials with integer coefficients. Suppose some prime p divides $\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}$ and μ_{n-1} but not μ_n nor μ_0/p. Show why $M(z)$ must be irreducible. (A classical problem treated in some Algebra texts.)

Proof 4a: The proof builds a contradiction. For the sake of argument suppose $M(z) = B(z)\cdot P(z)$ where $B(z) = \sum_{0\le k\le m} ß_k\cdot z^k$ and $P(z) = \sum_{0\le k\le n-m} \pi_k\cdot z^k$ with $1 \le m \le n-1$ and coefficients $ß_k$ and π_k all integers, whence $\mu_L = \sum_{\max\{0,\, L+m-n\} \le k \le \min\{L,\, m\}} ß_k\cdot\pi_{L-k}$ for $0 \le L \le n$.

Because $\mu_0 = ß_0\cdot\pi_0$ is divisible by p but not p^2, just one of $ß_0$ and π_0 would be divisible by p; for definiteness suppose it were $ß_0$ and not π_0. But p could not divide every $ß_k$ lest p also divide $\mu_n = ß_m\cdot\pi_{n-m}$ contrary to our problem's supposition. Let L be the least index for which p did not divide $ß_L$. Necessarily $1 \le L \le m$, so $ß_L\cdot\pi_0 = \mu_L - \sum_{L+m-n \le k \le L-1} ß_k\cdot\pi_{L-k}$; it would be divisible by p (as is every term in the right-hand side) if not for our contradictory suppositions about $ß_L$ and π_0. This contradiction proves that $M(z)$ must be irreducible.

Here is an alternative proof. We work in the *Field* $\mathbb{Z}_p$ of integers **mod** p consisting of *Residues* (remainders) $0, 1, 2, \dots, p-1$ obtained when integers are divided by the prime p. Let $\bar{\mu}_k$, $\bar{ß}_k$ and $\bar{\pi}_k$ be the residues when μ_k, $ß_k$ and π_k respectively are divided by p; these are written $\bar{\mu}_k := \mu_k$ **mod** p *etc*. Then $\overline{M}(z) := M(z)$ **mod** $p = \sum_{0\le k\le n} \bar{\mu}_k\cdot z^k$ and similarly for $\overline{B}(z)$ and $\overline{P}(z)$. Our problem's hypotheses imply that $\overline{M}(z) = \bar{\mu}_0\cdot z^n \ne 0$ and $\mu_n \ne 0$ **mod** p^2. Our supposition for the sake of argument that $M(z) = B(z)\cdot P(z)$ would imply that $\overline{B}(z)\cdot\overline{P}(z) = \overline{M}(z) = \bar{\mu}_0\cdot z^n$ whence would follow first that $\overline{ß}_m \ne 0$ and $\overline{\pi}_{n-m} \ne 0$ because $\overline{ß}_m \cdot \overline{\pi}_{n-m} = \overline{\mu}_n \ne 0$, and consequently that $\overline{B}(z) = \overline{ß}_m \cdot z^m$ and $\overline{P}(z) = \overline{\pi}_{n-m} \cdot z^{n-m}$ (work it out). These, by forcing $\overline{ß}_0 = \overline{\pi}_0 = 0$, would make $\mu_0 = ß_0 \cdot \pi_0 = 0$ **mod** p^2 contrary to the problem's last hypothesis. Therefore $M(z)$ is irreducible.

Note: These proofs work also for a polynomial $x^n \cdot M(1/x)$ whose coefficients are $M(z)$'s in reverse order.

Problem 4a's assertion is called "Eisenstein's Criterion for Irreducibility". It applies to some, not all irreducible polynomials. The only other scheme we know to determine whether an arbitrary polynomial is irreducible is to submit it to the factorization program in a computerized algebra system like *Maple*, *Macsyma*, *Mathematica*, *etc*. If none of them can factorize the polynomial, it is irreducible unless a bug not yet discovered lurks in their programs.

Problem 4b: Prove that, for every integer $n \ge 1$, there exist irreducible polynomials of degree n whose n zeros are all real. (Not so easy**!**)

Proof 4b: There are many such polynomials. We'll use *Chebyshev Polynomials* defined thus:

$T_0(x) := 1$, $T_1(x) := x$, and $T_{n+1}(x) := 2x \cdot T_n(x) - T_{n-1}(x)$ for $n = 1, 2, 3, \ldots$ in turn.

By induction we confirm that $T_n(x)$ is a polynomial of degree n whose coefficient of x^n is 2^{n-1} for $n \ge 1$, and $T_{2n}(0) = (-1)^n$, $T_{2n-1}(0) = 0$, and $T_n(x) = \cos(n \cdot \arccos(x))$ on $-1 \le x \le 1$.

As x runs down from 1 to -1 the value of $T_n(x)$ oscillates from 1 to -1 to 1 to -1 to … to $(-1)^n$, crossing through zero n times. Consequently the polynomials $W_n(x)$ defined by

$$W_{2n-1}(x) := 3 \cdot T_{2n-1}(x) - 1 \quad \text{and} \quad W_{2n}(x) := 3 \cdot T_{2n}(x) - 2 \cdot (-1)^n$$

have all zeros real and satisfy Eisenstein's Criterion (reversed) for Irreducibility with $p = 3$.

An alternative proof shows that the polynomials in question do exist without providing an explicit construction for any of them. First comes the following observation:

> **Lemma:** If $F(x)$ is a real polynomial of degree $n \ge 1$ whose n zeros are all real and distinct, then the same is true of $F(x) - \phi$ for each nonzero real constant ϕ with $|\phi|$ small enough.

Proof of the Lemma: Rolle's Theorem says that the derivative $F'(x)$ has $n-1$ real zeros y_j each between adjacent zeros of $F(x)$. Consequently $F(y_1)$, $F(y_2)$, … and $F(y_{n-1})$ are the $n-1$ local maxima of $|F(x)|$, with signs alternately positive and negative; each yj is located between two adjacent zeros of $F(x)$. Provided $|\mu| < \min_j |F(y_j)|$ the same is true of $F(x) - \mu$; its n real zeros straddle the same locations y_j where each $F(y_j) - \mu$ has the same sign as $F(y_j)$. End of proof.

The Lemma will be used to generate in turn polynomials $F_1(x)$, $F_2(x)$, $F_3(x)$, …, $F_n(x)$, … : each F_n will be irreducible with integer coefficients, have degree n, and have n real distinct nonzero zeros. We start with $F_1(x) := x - 3$, say. Next, after $F_n(x)$ has been determined for any integer $n \ge 1$ we construct $F_{n+1}(x) := p_n \cdot x \cdot F_n(x) - 1$ where p_n is a huge prime chosen thus:

Since all $n+1$ zeros of $x \cdot F_n(x)$ are real and distinct, the Lemma says some positive ϕ_n exists such that all $n+1$ zeros of $x \cdot F_n(x) - \phi$ are real and distinct (and nonzero) for every nonzero ϕ

with $|\phi| < \phi_n$. Choose prime p_n bigger than $1/\phi_n$ and also bigger than every prime divisor of the leading coefficient of x^n in $F_n(x)$. This is feasible because, as Euclid showed, there are infinitely many primes. Then all n+1 zeros of $F_{n+1}(x) := p_n \cdot x \cdot F_n(x) - 1$ are real, distinct and nonzero, and F_{n+1} satisfies Eisenstein's Criterion (reversed) for Irreducibility too.

This Problem 4b was taken from the Fall 2007 Prelim. Exam for Math. Grad Students.

Problem 5a: Suppose the plane is colored with two colors; in other words, suppose some points are red, say, and the rest blue. Must some two points an inch apart have the same color? Why?

Solution 5a: Yes; here is why: Consider the vertices of any equilateral triangle whose sides are one inch long. Among the three vertices are only two colors; two vertices must be colored alike.

Problem 5b: The same questions if the plane is colored with three colors instead of two. (Hard**!**)

Solution 5b: Yes, some two points an inch apart must be colored alike; here is how to find some: First examine any circle of radius $\sqrt{3}$ inches. If all points on the circle are colored alike, all of its chords one inch long join two points colored alike. Otherwise, some point(s) on the circle must be colored differently than its center. Suppose its center C is red, say, and a point P on the circle is green, say. Two circles of radius one inch centered at C and at P intersect at two points A and B each distant one inch from the other, from C, and from P. (Can you see why?) If A and B are colored differently, one of them must be colored the same as either C or P since there are at most three colors among the four points. Therefore some two of the four points A, B, C and P must be colored alike, and those two are not C and P. End of explanation.

What if the plane is colored with *four* colors instead of two or three? *Five* ? Nobody knows.

Problem 6: $\sum_{0\le k\le n} \cos(2k\cdot x) = \cos(n\cdot x)\cdot\sin((n+1)\cdot x)/\sin(x)$. Why? (Supply a short proof.)

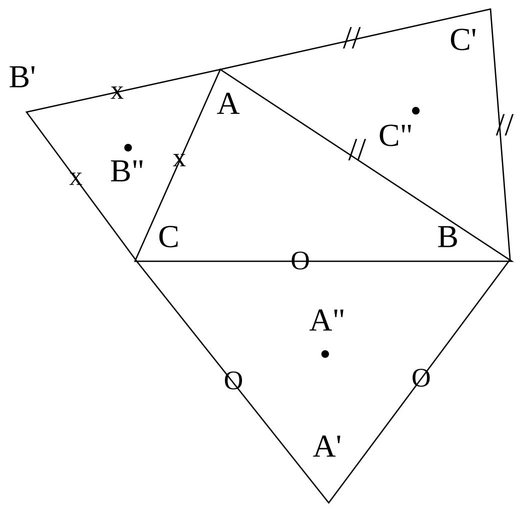
Proof 6: The trigonometric identity $\sin((2k+1)\cdot x) - \sin((2k-1)\cdot x) = 2\cdot\cos(2k\cdot x)\cdot\sin(x)$ turns twice the sum into $2\cdot\sum_{0\le k\le n} \cos(2k\cdot x) = \sum_{0\le k\le n} \big(\sin((2k+1)\cdot x) - \sin((2k-1)\cdot x)\big)/\sin(x)$ which first collapses into $2\cdot\sum_{0\le k\le n} \cos(2k\cdot x) = \big(\sin((2n+1)\cdot x) - \sin((0-1)\cdot x)\big)/\sin(x)$ and then becomes $2\cdot\sum_{0\le k\le n} \cos(2k\cdot x) = \big(\sin((n+1+n)\cdot x) + \sin((n+1-n)\cdot x)\big)/\sin(x) = 2\cdot\cos(n\cdot x)\big(\sin((n+1)\cdot x)/\sin(x)\big)$ to confirm the problem's assertion. If x is an integer multiple of π replace $(0/0)$ by n+1 .

An alternative proof uses the complex variable $z := e^{\imath x} = \cos(x) + \imath\cdot\sin(x)$ wherein $\imath := \sqrt{-1}$: Then $z^{2k} = \cos(2k\cdot x) + \imath\cdot\sin(2k\cdot x)$ and the sum in question turns into the real part of the finite geometrical series

$$\sum_{0\le k\le n} z^{2k} = (z^{2n+2} - 1)/(z^2 - 1) = z^n\cdot(z^{n+1} - z^{-n-1})/(z - z^{-1}) = z^n\cdot\sin((n+1)\cdot x)/\sin(x)$$

whose real part is $\cos(n\cdot x)\cdot\sin((n+1)\cdot x)/\sin(x)$, confirming the problem's assertion again.

Problem 7: Given an arbitrary non-degenerate triangle ABC, erect three equilateral triangles ABC', BCA' and CAB', one on each edge of ABC, with C and C' on opposite sides of AB, A and A' on opposite sides of BC, and B and B' on opposite sides of CA. Let C" be the center of ABC', A" the center of BCA', and B" the center of CAB'. Explain why A"B"C" must constitute a fourth equilateral triangle.



Proof 7: Matrix $R := \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}/2$ is *Orthogonal* ($R^{-1} = R^T$) and represents a rotation of the plane counter-clockwise through $2\pi/3$ radians ($120°$), so $R^3 - I = O$. This last equation factors into $(R - I)\cdot(R^2 + R + I) = O$, but $\det(R - I) = 3 \neq 0$, so $R^2 + R + I = O$. This last equation will let us eliminate $R^2 = -R - I$ from equations below.

Choose an origin **o** in the plane arbitrarily, and let **A** be the 2-vector that displaces **o** to vertex A. Do similarly for **A'**, **A"**, **B**, **B'**, **B"**, **C**, **C'** and **C"**. Evidently $\mathbf{C'} - \mathbf{B} = R\cdot(\mathbf{B} - \mathbf{A})$, so $\mathbf{C'} = (I + R)\cdot\mathbf{B} - R\cdot\mathbf{A}$. Similarly $\mathbf{A'} = (I + R)\cdot\mathbf{C} - R\cdot\mathbf{B}$ and $\mathbf{B'} = (I + R)\cdot\mathbf{C} - R\cdot\mathbf{B}$. Substitute these equations into the expressions for the centers $\mathbf{C"} = (\mathbf{A} + \mathbf{B} + \mathbf{C'})/3$, $\mathbf{A"} = (\mathbf{B} + \mathbf{C} + \mathbf{A'})/3$ and $\mathbf{B"} = (\mathbf{C} + \mathbf{A} + \mathbf{B'})/3$ to get $3\cdot\mathbf{C"} = (I - R)\cdot\mathbf{A} + (2I + R)\cdot\mathbf{B}$, $3\cdot\mathbf{A"} = (I - R)\cdot\mathbf{B} + (2I + R)\cdot\mathbf{C}$ and $3\cdot\mathbf{B"} = (I - R)\cdot\mathbf{C} + (2I + R)\cdot\mathbf{A}$. Note these equations' rotational symmetry $A \to B \to C \to A$ which yields two more equations from any one of them, thus diminishing the algebraic work.

Now the edge-vectors of triangle A"B"C" will be computed:

$$3\cdot(\mathbf{C"} - \mathbf{A"}) = (I - R)\cdot\mathbf{A} + (I + 2R)\cdot\mathbf{B} - (2I + R)\cdot\mathbf{C} = 3R\cdot(\mathbf{B"} - \mathbf{C"}) ;$$
$$3\cdot(\mathbf{A"} - \mathbf{B"}) = -(2I + R)\cdot\mathbf{A} + (I - R)\cdot\mathbf{B} + (I + 2R)\cdot\mathbf{C} = 3R\cdot(\mathbf{C"} - \mathbf{A"}) ; \quad \text{and}$$
$$3\cdot(\mathbf{B"} - \mathbf{C"}) = (I + 2R)\cdot\mathbf{A} - (2I + R)\cdot\mathbf{B} + (I - R)\cdot\mathbf{C} = 3R\cdot(\mathbf{A"} - \mathbf{B"}) .$$

The rightmost three equations are confirmed by simplification after substituting $-R - I$ for R^2. They say that each edge of A"B"C" is obtained by rotating a neighboring edge through $2\pi/3$, whence A"B"C" must be equilateral, as problem **7** claimed.

By the way, Problem 7's claim is valid also when ABC is a degenerate triangle but not just a single point.