Computation of Optimal Break Point Set of Relays—An Integer Linear Programming Approach

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***Abstract*—We propose an integer linear programming (ILP) formulation for the minimum relay break point set (BPS) computation. Subsequently, in the ILP framework, we propose an alternate maximum-independent relay BPS formulation with the intention of minimizing dependency within the BPS. We show that 1) in practice, the relaxed version of the ILP suffices to obtain an integral vertex and 2) the relaxed version of the ILP can be efficiently solved by the dual-simplex method. The performance of the proposed algorithm is compared and contrasted with existing algorithms. Case studies on various test systems show the efficacy of the proposed approach.**

***Index Terms*—Greedy algorithms, integer linear programming, minimum break point set (MBPS), NP-complete problem.**

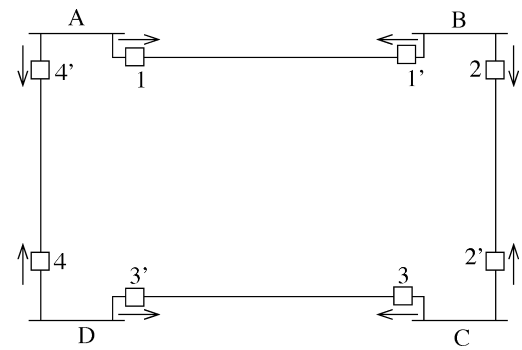

Fig. 1. Four-bus single-loop system.

I. INTRODUCTION

WE consider the problem of directional relay coordination in a meshed system. The directional relay coordination problem becomes iterative in nature whenever the system has a simple loop. To see this, let us consider a simple loop system shown in Fig. 1. Suppose we first set the relay 1. Then, the sequence of the relay setting for the clockwise loop is $1 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ (to be read from left to right). Since, the setting of relay 1 may be disturbed in the last step because of coordination requirements with relay 2, the process of relay setting may have to be repeated until there is no perceivable change in the setting of relay 1. In this way, the coordination problem for the system with simple loop(s) becomes iterative. To design a "good" relay coordination procedure, one must choose an initial set of relays to be set with care. Relays in this set should satisfy desirable conditions as follows.

1) For each directional coordination loop, there must be at least one relay in this set which is incident on this loop. This ensures that all directional loops will be coordinated. Since opening of these relays opens all directed simple loops, this set is called a break point set (BPS).
2) The BPS should be minimal (i.e., no proper subset of the BPS should be a BPS itself). The minimum cardinality BPS is known as the minimum break point set (MBPS). In general, multiple BPSs may satisfy the MBPS criterion. If the initial relay set corresponds to MBPS, then the convergence property of the directional relay coordination algorithm improves significantly. Hence, considerable research has been carried out to compute the MBPS.
3) If the BPS is chosen as per criteria 1) and 2), it is still possible that there may be dependency in the relay settings of the MBPS. This dependency should be minimized. Hence, we also desire a maximum independent, minimal (preferably minimum) BPS. While criteria 1) and 2) are known in the existing state of art, criterion 3) is proposed for the first time in this paper.

The idea of BPS in relay coordination was first introduced in [1]. Reference [2] is a comprehensive technical report on this issue, detailing ways to enumerate all directed loops in a network, finding a minimal BPS, relay sequence matrix[1] and sequential primary backup pairs. In this method, BPS is chosen as follows: Let S_i be the set of relays incident on the ith directed loop of the graph. Let Y_j be the set of loop indices in which the relay j participates. Let weight vector W be a vector such that $W_i = \sum_{k \in Y_i} |S_k|$. Now, to choose a relay k to be added to the BPS, we first choose a loop i with minimum $|S_i|$ and then choose a relay $k \in S_i$ with maximum W_k. All of the loops j such that $j \in Y_k$ are opened. For all i, S_i and for all j, Y_j are updated. This process is repeated until all of the loops have been opened.

References [3]–[5] also discuss approaches on similar lines. Their approach is an improvement over the graph-theoretic approach by Dwarakanath and Nowitz [6]. Some of the inefficiencies in the earlier approach of [2] have been addressed in [7]–[9]. Rao in [7] used the notion of loop matrix L_D.[2] In [7], the MBPS problem was mapped to find the minimum cover of relays such that their span is the whole set of directed loops. However, the minimum cover problem is the set-cover problem which is known to be NP complete [10]. Using Boolean functions and Boolean algebraic manipulations, the authors in [7] manage to obtain the minimum BPS. At the same time, an ex-

[1]A matrix which contains the order in which relay settings should be done.

[2]The columns stand for relays and the rows are for directed loops.

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ponential time computational effort (which is expected as the problem to which the MBPS problem was mapped is NP complete) is required. This is in contrast to earlier approaches such as [2] where a minimal and not a minimum set was obtained. This paper also gave a simplistic way of finding all of the simple loops. The authors in [3] have suggested the use of depth first search in order to enumerate all of the loops. In [8], it is reported that the breadth first search can further enhance the efficiency of finding all of the loops. The methodology developed in [9] does not require generation of all the directed loops. In [11], the authors have applied the concept of functional dependency to the topological analysis of the graph. This did not require the computation of all possible loops in the circuit and also did not require exponential time but settled for suboptimal solutions. This approach is extended by Madani and Rijanto in [12] and [13]. In [14], Madani and Rijanto have tried to improve their algorithm by using the concept of partitioning graphs into the forest.[3] Jamali and Shateri in [15] have tried to take advantage of the network decomposition ideas introduced in [14] and take a polynomial time approach but can end with rather suboptimal results. It may be noted that most of the approaches except [6], [7], and [9] do not give the optimum set but rather a suboptimal one. Sastry in [16] has tried a heuristic approach where a bus with the highest degree is chosen and all of the relays on that bus are removed. This process is carried out until no more loops are left in the system. Though the proposed algorithm is polynomial time, the solution is not guaranteed to be exact. The authors in [17] have also developed a heuristic, recognizing that the problem may be NP-hard. Their approach is as follows:

At each step, a bus, which participates in maximum number of coordination loops, is chosen. Subsequently, all of the relays on that bus are selected in the BPS and the relevant coordination loops are opened. The procedure is repeated until there are no coordination loops left in the network.

Recently, in [18], Yue *et al.* tried to circumvent the problem of enumerating all of the loops and solving the problem in polynomial time but ended up with suboptimal results. Their method is as follows.

Let P_i be the set of primary relays for the relay i. This set is referred to as its primary relay dependency set (PRDS). Its cardinality is referred to as the primary relay dependency dimension (PRDD). Now, choose a relay k with maximum PRDD and add it to an MBPS. Thereafter, this relay is removed from all of the sets. In this process, if the PRDS associated with any other relay also reduces to the null set, then the process is repeated with this relay also. However, this relay is not added to the BPS. This process is repeated until all sets are reduced to null set.

As is evident, their approach is a greedy algorithm. It may be noted that the approach in [17] is also a greedy approach but gives much better results than [18]. On the other hand, it has to pay the price of being an exponential time algorithm compared to the approach in [18], which is a polynomial time algorithm.

In this paper, we formulate the following optimal BPS computation problem as an ILP.

- MBPS computation problem;
- maximum-independent minimal BPS computation problem.

[3]By partitioning a graph into the forest, we mean finding a partition of the vertex set such that the subgraph induced by each partition is a tree.

Subsequently, algorithms such as the cutting plane algorithm and branch-and-bound algorithm are used for solving this NP-hard problem (ILP is NP-hard). The proposed approach guarantees an optimal solution. As ILP itself is NP hard, the problem may require exponential time in the worst-case scenario. Therefore, for large systems, we also develop a decomposition approach. The proposed algorithm has shown extremely good performance.

It is assumed throughout this paper that we are dealing with systems where every relay is incident on at least one loop. Otherwise, such relays may be eliminated in the preprocessing step and such an elimination neither affects the cardinality of the MBPS nor the maximum-independent minimal BPS. It is further assumed that the network we are dealing with is a connected one. In case the network is not connected, we can separate the components of the network in the preprocessing step and then deal separately with the components. Throughout this paper, the term "optimal BP" stands for both the MBPS as well as maximum-independent minimal BPS.

This paper is organized as follows. Section II formulates the optimal BPS problem as an ILP. Section III describes the proposed algorithm. An illustrative example is worked out in Section IV. Section V deals with techniques required for large-scale implementation. Section VI shows the results and analysis of various algorithms. Section VII discusses certain pathological cases or exceptions to the generic behavior and Section VIII concludes this paper.

II. Optimal BPS Computation as an ILP

This section details the following two important contributions of this paper.

- Development of ILP framework for solving the well-known MBPS problem.
- Improvement in the objective function from the computation of MBPS to maximum-independent minimal BPS.

We begin with the formulation of the MBPS problem as an ILP.

A. MBPS Formulation as an ILP

Let the matrix A be the same as matrix L_D as discussed in [2]. We formally define it as a matrix where rows represent directed loops, and columns represent relays. Matrix entry $A_{ij} = 1$ if the jth relay is incident on the ith directed loop; otherwise, $A_{ij} = 0$. Let us first define a Boolean set

$$\mathbb{B} = \{0, 1\}. \tag{1}$$

If m denotes the number of rows in A, and n denotes number of columns in A, then

$$A \in \mathbb{B}^{m \times n}. \tag{2}$$

The problem of finding the minimum BPS can now be formulated as follows: Find a minimum set of relays such that each directed loop has at least one of the chosen relays. Let us associate a variable x_i with each relay i which takes a value of 1 if the corresponding relay is chosen in the BPS and 0 if it is not. We now associate a characteristic vector with the set of relays $x = [x_1 x_2 \ldots x_n]^t$. Since each directed loop must involve at least one of the relays, therefore, for all i, $a_i^t x \geq 1$, where a_i^t

stands for the ith row of the matrix A. Let b denote a vector, all of whose entries are 1 and $b \in \mathbb{B}^m$. Then, a feasible set of solutions to (3), contains all possible BPS where BPS corresponds to relay(s) with $x_i = 1$ in a feasible x vector

$$Ax \geq b \quad x \in \mathbb{B}^n. \tag{3}$$

To achieve MBPS, we use a linear cost function with the unit cost for each relay. Let c be a vector with all of the entries being 1 and $c \in \mathbb{B}^n$. So the minimum cardinality condition is given by

$$\min c^T x. \tag{4}$$

The minimization problem defined in (4), along with the constraint set (3), maps the problem of finding MBPS to a binary LP problem. Let $\mathbb{Z}$ denote the set of integers. Then, the constraint (3) can be rewritten as

$$Ax \geq b \quad \text{for all i} \quad 0 \leq x_i \leq 1 \quad x \in \mathbb{Z}^n. \tag{5}$$

Now (4) and (5) define an ILP problem. Further, as shown in Proposition II.1, the upper bound on x can be dropped.

Proposition II.1: The upper bound on x defined by (5) is redundant.

Proof: Let x be an optimal solution to the ILP. Let us assume that i exists such that $x_i \geq 2$. Then, for all loops j such that relay i is incident on loop j, $a_j^t x \geq 2$. Construct a new vector $\tilde{x}$ in which all of the entries are the same as that of x, except $\tilde{x}_i = 1$. Clearly, $\tilde{x}$ is a feasible solution but the cost of $\tilde{x}$ shall be strictly lower than x. This is a contradiction to the fact that x is an optimal solution. Hence, at the optimal, for all i, $x_i \leq 1$. Hence, we can drop the upper bound on x in (5), and (4) and (6) formulate the mathematical model of the problem

$$Ax \geq b \quad x \in \mathbb{Z}^n \quad x \geq 0. \tag{6}$$

□

B. Maximum-Independent Minimal BPS Problem as an ILP

Given any minimal BPS solution, if $a_i^t x = 1$ for all i, then it implies that every directional loop has exactly one relay from the minimal BPS. This means that the settings of the relays in the minimal BPS are independent of each other. Conversely, if for some i, $a_i^t x > 1$, then it indicates that there is a coordination requirement among the relays in the minimal BPS. Hence, $\sum_{i=1}^m (a_i^t x - 1)$ is a measure of the dependency among the relays in the BPS. Therefore, we propose the following formulation for the maximum-independent minimal BPS problem:

$$\min \sum_{i=1}^m a_i^T x \tag{7}$$

subject to the constraints given by (6). We have excluded -1 from the objective function as it is a constant for all vectors x. Notice that the objective function in (7) may not lead to the MBPS. Conversely, an objective for MBPS computation may not lead to a maximum-independent minimal BPS. A comparative evaluation of the two objective functions is reported in Section VI.

In order to get the "goodness" of both the approaches, a composite objective function may be used. To get an idea of a composite objective function, let us consider the following function:

$$\min \lambda \sum_{i=1}^m a_i^T x + \eta c^T x. \tag{8}$$

Here, c stands for the same vector as in (4). The weights λ and η should be assigned according to the importance one wants to give to the maximum independence and minimum cardinality, respectively. If $\lambda \gg \eta$, then the obtained BPS shall be the BPS of minimum cardinality among all BPS of maximum independence. Similarly, if $\lambda \ll \eta$, then the obtained BPS shall be the maximum-independent MBPS.

III. Algorithm

Methods such as the cutting plane algorithm and branch-and-bound algorithm can now be used to solve the problem [19], [20]. Both of these methods solve relaxed LPs[4] to converge to the solution. By proposition II.1, we know that if there is an integral solution to the relaxed LP, then it must be on one of the vertices of the polytope. Hence, it is likely that the simplex algorithm, which moves from vertex to vertex of the polytope, may directly converge to an integral solution. Except for a pathological exception (discussed in Section VII), this behavior was consistently observed. On the other hand, we noticed that interior point methods, such as Karmarkar's algorithm [21], need not converge to an integral point. To see why this may occur, let us look at the following proposition.

Proposition III.1: Consider ILP framework for the MBPS problem. Let $\tilde{x}$ and $\hat{x}$ be two integral solutions of the ILP which are also optimal for the relaxed LP, then there are optimum solutions to the relaxed LP which are not integers.

Proof: Let $\grave{x} = \lambda\tilde{x} + (1-\lambda)\hat{x}$ where λ is an irrational number and $\lambda \in (0,1)$. Since λ is irrational, therefore $\grave{x}$ is certainly not an integral point. Clearly, $\grave{x}$ is a feasible solution as $A\grave{x} = (\lambda A\tilde{x} + (1-\lambda)A\hat{x}) \geq (\lambda b + (1-\lambda)b) = b$ and $\grave{x} \geq 0$. Let the optimum cost of the LP be f. Then, the cost of $\grave{x}$ is $\lambda f + (1-\lambda)f = f$. Hence, $\grave{x}$ is an optimum point though not an integral point. □

A. Dual Simplex vis-a-vis Simplex Algorithm

Further, our analysis showed that because of the structural properties of the matrix A, vectors b and c and the constraint set having only inequalities, it is preferable to use the dual-simplex method. In practice, we have seen that this can provide a speed up of up to 10 times. This is because phase I of the simplex method can be time consuming. Rewriting (4) and (6) in standard form (i.e., by putting slack variables), we have the transformed problem

$$\min \tilde{c}^T \tilde{x} \tag{9}$$
$$\widetilde{A}\tilde{x} = b \tag{10}$$
$$\tilde{x} \geq 0 \quad \tilde{x} \in \mathbb{Z}^{m+n} \tag{11}$$

[4]that is, LP is the same as the ILP except the constraint $x_i \in \mathbb{Z}$ is replaced by $x_i \in \mathbb{R}$ where $\mathbb{R}$ denotes the set of real numbers.

where

$$\tilde{c} = [c|0] \tag{12}$$

$$\widetilde{A} = [A| - I_m] \tag{13}$$

$$\tilde{x} = [x|x_s]. \tag{14}$$

Here $-I_m$ means the identity matrix of size m with a negative sign and the 0 in (12) means that a zero vector of dimension m is augmented to c. The variables x_s in (14) stand for slack variables. If we choose the last m columns as the columns of the initial basis, it can be observed that we obtain a primal infeasible solution. However, the corresponding choice on the dual problem leads to a feasible solution for the dual problem. Thus, the dual-simplex method should be preferred.

B. Handling of Phantom Buses

In transmission system protection, phantom buses are often present. Therefore, a realistic MBPS (or for that matter, any BPS) should exclude relays looking away from phantom buses (phantom relays). The formulation shown in the last section does not distinguish between the phantom and real buses. This may result in inclusion of phantom relays in the MBPS. To avoid this, the following method is used.

1) Elimination of Columns: In this technique, after formulating the problem in terms of A, b, and c, we removed the columns of A, and the entries in c that correspond to phantom relays. The corresponding entries of x should also be removed. The remaining entries of x will correspond to real relays. The reason for this step is that all of the simple directed loops should be broken (even those involving phantom buses), but every loop should be broken by the virtue of having at least one real participating relay in the MBPS. Hence, the inner product of every row of A with x should be bigger than 1, but the elements of x that are 1 should only correspond to real relays. After finding the modified A and c, the process of transforming the ILP from canonical to standard form may be carried out as explained before.

Remark 1: For simplicity, discussion in this section made specific reference to the MBPS problem. However, it also holds good for the maximum-independent minimal BPS problem.

IV. Illustrative Example

As an illustrative example, we consider the system shown in Fig. 2(a). The corresponding graph in Fig. 2(b) is a multigraph. Among the two edges between vertices H and E, let us call one of them as HE and the other as EH. There are 12 directed loops in this system. Hence, the vector b in this case shall consist of 12 entries

$$\mathbf{b^t} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1].$$

Similarly, the vector c shall consist of 14 entries as the number of relays in this network is 14 (number of edges in the network is 7). However, three of the relays (looking away from K) are phantom relays. Therefore, we eliminate the columns corresponding to the phantom relays. Hence, the final vector c will only consist of 11 entries. The final c is shown

$$\mathbf{c^t} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1].$$

It implies that the augmented vector $\tilde{c}$ shall have $11 + 12 = 23$ entries where the last 12 entries will be 0. Let us now name

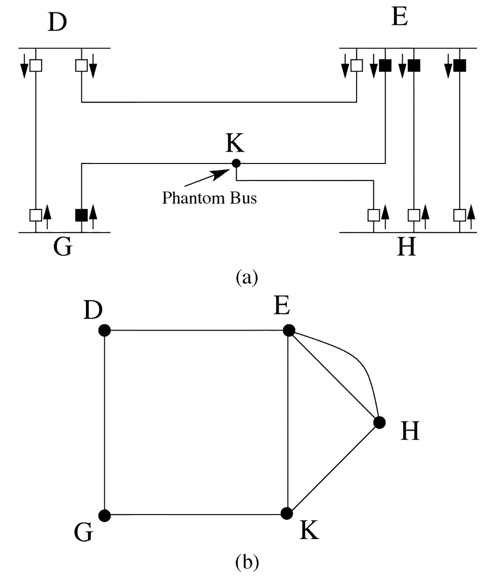

Fig. 2. Example system for demonstration [2, pg. 5–5]. (a) Example five-bus system. (b) Equivalent graph.

TABLE I
ALL RELAYS IN THE GRAPH SHOWN IN FIG.2(a)

Relay	Node	Edge
1	E	EK
2	D	DE
3	E	EH
4	G	DG
5	E	HE
6	G	KG
7	H	KH
8	E	DE
9	H	EH
10	D	DG
11	H	HE

the relays as shown in Table I. Here, the node indicates the bus from which the relay looks away and the edge means the line to which the relay belongs to. The matrix $\widetilde{A}^t$ with the first 11 rows representing the real relays according to the numbers assigned in Table I is shown in the equation at the bottom of the next page. One can start the dual-simplex method with the last 12 columns of the $\widetilde{A}$ matrix in basis (i.e., choosing the origin as the vertex of the dual problem). Hence, we will start with a primal infeasible but a dual feasible solution. After a few iterations, the algorithm converges to an integer solution

$$\mathbf{x^t} = (1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0).$$

The MBPS has been indicated (blackened relays) in Fig. 2(a). In order to find the maximum-independent minimal BPS, we need to change the cost function. The cost function is a vector where the entry corresponding to each relay is equal to the number of directed loops it participates in. Therefore

$$\mathbf{c^t} = (3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 4 \quad 3 \quad 3 \quad 3 \quad 3).$$

Using this as the cost function and $\widetilde{A}$, b, which is the same as that used in the MBPS computation, we solve the ILP and arrive at the integer vertex

$$\mathbf{x^t} = (1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0).$$

It can be seen that in this case, the chosen MBPS is itself a maximum-independent minimal BPS.

It is noteworthy that without the addition of any constraints (such as Gomory cuts in the case of the cutting plane method, which is usually required for solving an instance of ILP), the solution for MBPS as well as maximum-independent minimal BPS converges directly to an integral solution. It may occur in the general case that the solution to the relaxed LP is not integral. Even in that case, the cost of the relaxed LP does provide a lower bound on the cost of the ILP and, hence, a lower bound on the cardinality of the MBPS. In the next section, we discuss methods for handling large systems.

V. Handling Large Systems

In case of large-scale systems, the price one has to pay for the exactness of the solution is that the computational space and time requirements may shoot up exponentially. This is not because simplex is being used to solve the LP or due to the addition of constraints (for example, Gomory cuts in the cutting plane method) but because enumeration of all the loops requires an exponential effort. Therefore, in case the network is very large, one may try to solve the problem simply by exploiting the decompositions in the network arising out of natural or administrative reasons. For example, the power grid in Western India is a relatively large network but there are few interconnections between any two states. So a natural decomposition will be to solve the system individually for each state and then augmenting necessary relays in order to form the BPS of the whole network. Once we have obtained a decomposition of the system and obtained an optimal BPS for each subsystem, the following two procedures can be used to obtain a minimal BPS for the system.

A. Approach 1

1) Arbitrarily assign a distinct number to each individual subsystem.
2) For each pair of subsystems (i, j) and $i < j$, add all of the relays that are on edges between i and j and facing away from i to the BPS.

Referring to the block diagram in Fig. 3, if one assumes that the optimal BPS in the networks M_1, M_2, and M_3 have been obtained, then augment all of the relays on edges between M_1 and M_2 which face away from M_1 to the BPS obtained from subsystems M_1, M_2, M_3. Similarly, in the next step, all of the relays that are between M_2 and M_3 which face away from M_2 should be added to the BPS. Finally, all of the relays that are between M_1 and M_3, which face away from M_1, should be added to the BPS to obtain the final BPS. The relays to be removed on the bridges have been indicated (blackened relays) in Fig. 3.

B. Approach 2

There is also another way of finding relays between subsystems which should be augmented to the optimal BPS of each subsystem, so as to obtain a minimal BPS of the whole network. Let us assume that we have decomposed the network into subsystems $M_1, M_2, \ldots, M_k$. Associate with each of them, a supernode and then form a reduced network wherein the lines between M_i and M_j in the original network are retained as lines

$$\tilde{A}^t = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

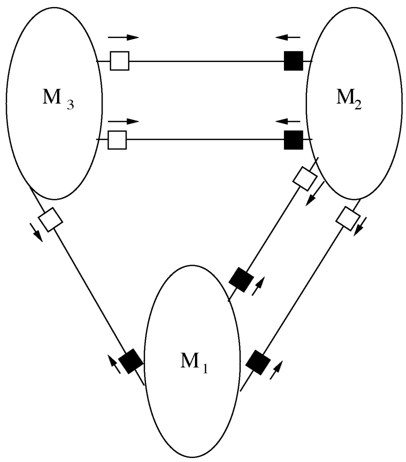

Fig. 3. Block diagram of a separable large network—1st augmenting technique.

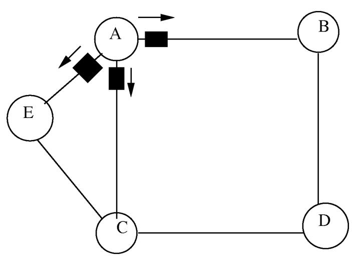

Fig. 4. Block diagram of a separable large network—2nd augmenting technique.

between the corresponding supernodes in the new network. Subsequently, the ILP algorithm is run on this reduced network and the BPS is obtained so that it can be augmented to the BPS of the subsystems in order to obtain a minimal BPS of the original network. The minimal BPS obtained by this technique will be at least as good as the one obtained by approach I. Referring to Fig. 4, assume that A, B, C, D, and E represent subnetworks. If we run the MBPS algorithm over a new graph assuming A, B, C, D, and E as buses and the interconnections as the lines, we obtain an MBPS of cardinality 3. One such set is indicated in the figure. This set of relays should be augmented to the MBPS of different subnetworks in order to obtain a minimal BPS. It is obvious that if approach I was adopted, then the size of the set of relays to be augmented would have been 6.

C. Handling Parallel Lines

In systems such as the western regional grid of India (Fig. 11), one observes three to four parallel 400-kV lines emanating from large power stations. This can cause a steep increase in the number of loops in the system. One way in which this problem can be circumvented is that we may remove certain parallel edges in the network and then try to find the solution to the reduced problem. Once the optimal BPS for the reduced problem has been obtained, the following procedure may be adopted for augmenting this BPS in order to form a minimal BPS of the original problem. For this, let us consider any edge e between buses a and b in the reduced graph and the set of parallel edges E to it in the original graph which have been removed (the set E may be empty). In case, the set E is nonempty and

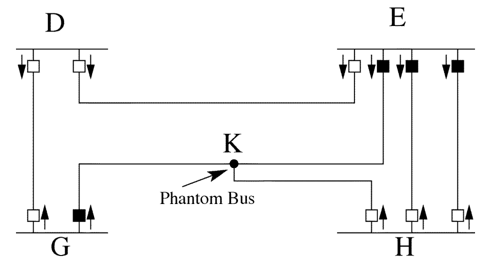

Fig. 5. Sample five-bus system.

TABLE II
RESULTS FOR THE FIVE-BUS SYSTEM SHOWN IN FIG. 5

Algorithm	Cardinality of BPS
Proposed Algorithm	**4**
Algorithm in [17]	4
Algorithm in [2]	6
Algorithm in [18]	4

1) the optimal BPS for the reduced problem does not have any relay from edge e, then augment this BPS, with relays on E and e which face away from a (instead of a, b can also be chosen).
2) In case the reduced BPS has both relays on e, then augment all of the relays on E to the reduced BPS.
3) If only one relay (say one that faces away from a) is included, then augment all of the remaining relays on E, which face away from a to the reduced BPS.
4) Repeat this process for each edge e in the reduced graph for which the corresponding E is nonempty.

The BPS so obtained may not be optimal but will certainly be minimal. Considering that in the worst case, computing the optimal BPS becomes an enumerative problem, a slightly suboptimal solution is acceptable to us.

VI. Results and Comparisons

In this section, results for the algorithm for MBPS computation as well as maximum-independent minimal BPS computation are discussed. We begin with results for MBPS computation.

A. Results for MBPS Computation

For evaluating the performance of the proposed algorithm, an efficient code for ILP was written in JAVA which was tailored for solving sparse systems. Diagrams for various systems are shown in Figs. 5–11. Results for the various test systems are shown in Tables II–X. The tables indicate the results using algorithms which compute only a minimal and not necessarily a minimum set. It was found that the proposed algorithm outperformed each of these methods. The cardinality of the MBPS computed from the proposed approach will be identical to that obtained by other exact methods [7], [9]. This was also confirmed by implementation. However, it was found that the proposed method outperforms other exact methods in terms of computational effort. This issue is addressed in the latter part of the section.

1) 5 Bus System: The five-node system, which is discussed in [2], is shown in Fig. 5. The results using algorithms developed in [2], [17], [18] along with the proposed algorithm are presented in Table II. It can be seen that other than the method

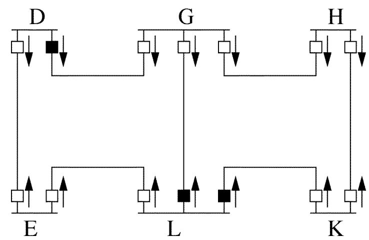

Fig. 6. Sample six-bus system.

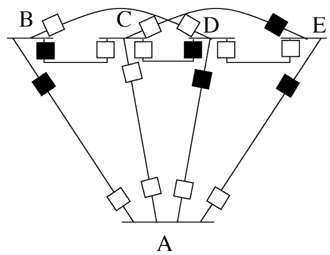

Fig. 7. Symmetric five-bus system.

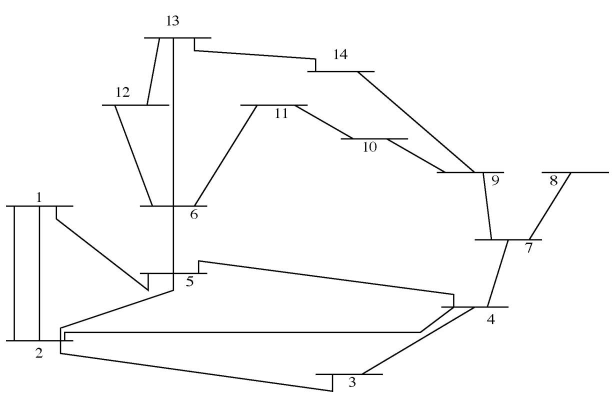

Fig. 8. Topology of the IEEE 14-bus system.

discussed in [2], every other method led to the optimum answer. In this case, the dual-simplex method provided an integral solution to the relaxed LP without adding any more constraints. The MBPS obtained by the proposed approach is indicated in Fig. 5 (blackened relays).

2) Six-Bus System: The six-node system discussed in [2] is shown in Fig. 6. In this case, each method led to the optimum answer which is shown in Table III. As in the previous case, the solution to the relaxed LP came out to be integral. The MBPS obtained by the proposed algorithm is marked (blackened relays) in Fig. 6.

3) Five-Bus Symmetric System: The system is shown in Fig. 7 and the results are shown in Table IV. It can be seen that the proposed algorithm performs much better than the algorithm in [18]. Again here, the solution to the relaxed LP converged to an integral point. The blackened relays in Fig. 7 indicate the MBPS obtained by the proposed approach. Notice that the relay on edge EC, which faces away from E, has to be coordinated with the relay on edge AB which faces away from B. This shows that relays in MBPS can have interdependency among them.

4) IEEE 14-Bus System: The topology of the IEEE 14-bus system is shown in Fig. 8. The results are summarized in Table V. It can be seen here that the optimum answer is obtained only with the proposed approach. As has been in the previous cases, the solution to the relaxed LP converged to an integral point without adding any more constraint. It was found out that there are directed loops in the system in which more than one relay from the MBPS participates in. This indicates that there is interdependency among the relays in the MBPS.

5) IEEE 30-Bus System: The topology of IEEE 30-bus system is shown in Fig. 9. The results are shown in Table VI. The dual-simplex method again gave an integral solution to the relaxed LP, without adding any more constraint. Again, a considerable amount of interdependency was observed among the relays in the MBPS.

6) Indian Utility System: The Indian utility system is shown in Fig. 10. It has nine buses, 20 lines, and 1190 directed loops. The results are shown in Table VII. For this system also, the solution to the relaxed LP was integral. As in the previous cases, there were relays in the MBPS whose settings need to be coordinated with those of other relays in the MBPS.

7) Western Power Grid 400 kV in India: The 400-kV Western power grid in India is a large system which is shown in Fig. 11. The term "large" is used here to emphasize on the fact that the number of simple loops blows out. This is because the system has 49 fundamental loops. It implies that enumeration of all possible loops will require computation of 2^{49} linear combinations of the fundamental loops. This number is on the order of 10^{14} and, hence, enumeration of loops is computationally intractable. The system has 52 buses and 100 lines. There are five states in this grid namely Maharashtra, Madhya Pradesh, Chattisgarh, Gujarat, and Goa. The number of buses and lines in each state is indicated in Table VIII. The remaining 15 edges are interconnections between various states. The MBPS of the networks in different states was individually found out and then relays were augmented to the reduced BPS, as explained in Approach 1 (Section V). The results for the system are indicated in Table IX. The minimal BPS was obtained by augmenting the MBPS of each state with the minimal BPS on the interconnections and its cardinality was found out to be 72. It was found out that even if the BPS algorithm is run with the states as "supernodes" and the interconnections between them as lines (approach 2), then also the BPS to be augmented had cardinality equal to 15. This is equal to the number of interconnections. The lower bound for MBPS is 50, while the upper bound is 98. This is derived from the fact that the cardinality of the MBPS is at least one more than the number of fundamental loops and, at most, twice the number of fundamental loops. It comes as a nice surprise that even in such a large system, the relaxed LP for each state had an integral solution and the dual-simplex method converged to the same for each of them.

It was observed that for the large systems of Maharashtra and Madhya Pradesh, there was significant interdependency among the relays in the MBPS. The MBPS for Gujarat had no interdependency while that for Chattisgarh had a very small interdependency. The system for Goa, which is trivial, had no MBPS.

B. Comparison With Exact Methods

The proposed approach was compared with the methods developed in [7], [9]. For the five-bus system (Fig. 5), the six-bus

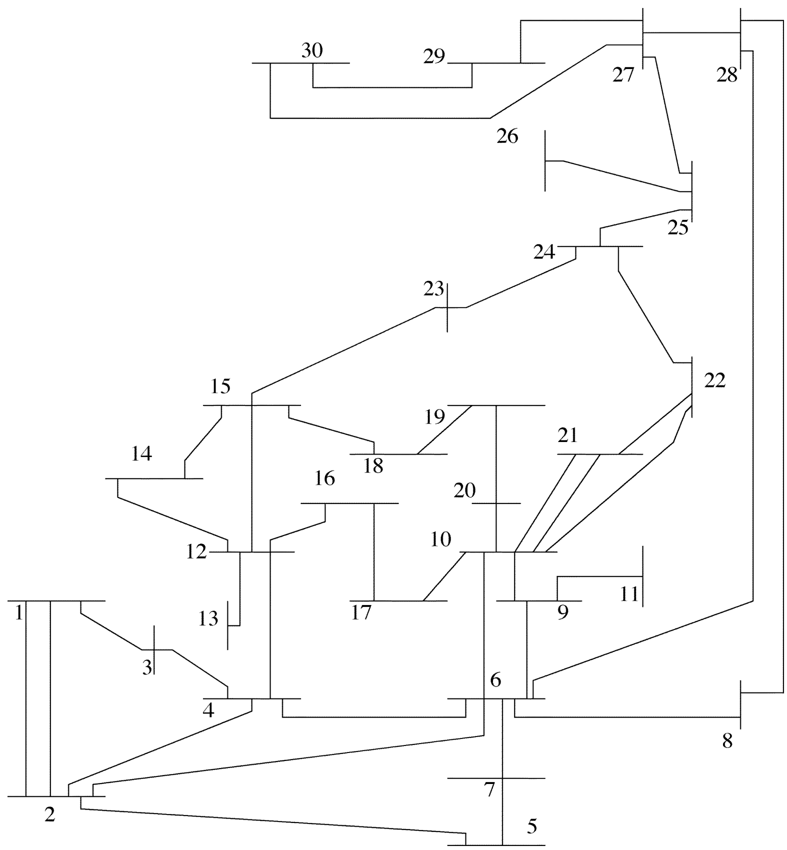

Fig. 9. Topology of the IEEE 30-bus system.

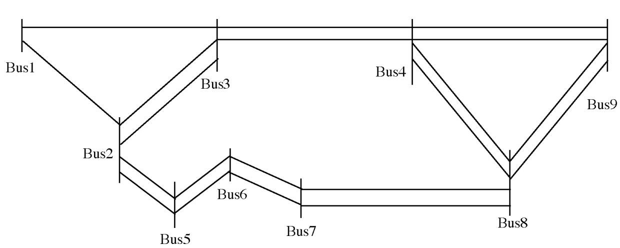

Fig. 10. Topology of the Indian utility system.

system (Fig. 6) and the symmetric five-node system (Fig. 7), the same cardinality of BPS was obtained as with the proposed ILP approach. There was no significant difference in computation time. However, for the IEEE 14-bus system, the approach in [7] took close to 5 h of computation time. The reason for this explosion in computation effort can be explained as follows:

The method in [7] forms Boolean expressions (which are sum of literals) corresponding to every directed relay coordination loop. Subsequently, these Boolean expressions are multiplied and simplified to obtain a Boolean expression representing the set of all possible minimal BPS. This expression is in sum-of-products form. In this expression, each disjunct represents a BPS of the whole system. Table X indicates the lower bound on a number of minimal BPS for test systems. This lower bound was calculated by counting the number of distinct trees in the graph (further discussed in the Appendix). As can be seen from the table, most of the lower bounds are exorbitantly large. This suggests that it is not the data structure or implementation but rather the inherent intractability of the method that makes it infeasible to be scaled up for large systems. Let us consider the following example. Consider a directed loop which has three participating relays, namely a, b, and c. The Boolean expression representing the possible BPSs for this single directed loop is

$$a + b + c$$

Let there be another coordination loop which has three relays, namely c, d, and e. The Boolean expression representing the possible break point sets for this loop is

$$c + d + e$$

If both of these loops are to be opened, then the Boolean expression for the set of minimal BPS of these two loops is

$$(a + b + c)(c + d + e) = c + ad + bd + ae + be.$$

This clearly requires $3 \times 3 = 9$ steps to compute the sum-of-products expression and further steps to simplify it.

In contrast, the ILP approach only has a worst-case exponential behavior. However, as observed earlier, relaxed LP itself suffices. Computational technology for solving linear programming is quite sophisticated and matured. The proposed ILP approach has the advantage of reaping its benefits. Since the method developed in [9] is similar to that in [7], therefore similar remarks hold good for [9]. The reasons previously mentioned made it impractical to even try these methods for larger systems

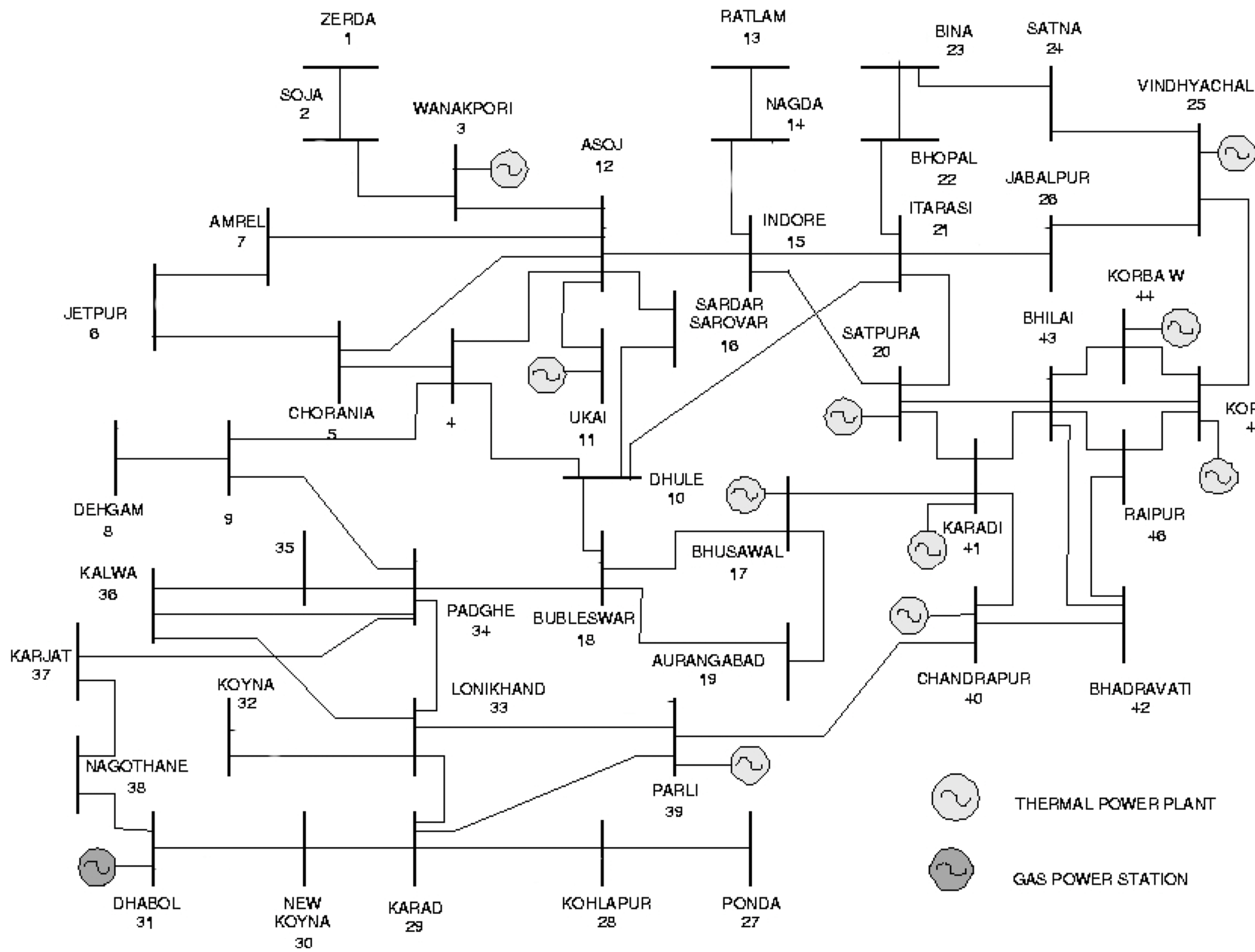

Fig. 11. Topology of 400-kV network of Western Power Grid, India.

TABLE III
RESULTS FOR THE SIX-BUS SYSTEM SHOWN IN FIG. 6

Algorithm	Cardinality of BPS
Proposed Algorithm	**3**
Algorithm in [17]	3
Algorithm in [2]	3
Algorithm in [18]	3

TABLE IV
RESULTS FOR THE FIVE-NODE SYMMETRIC SYSTEM SHOWN IN FIG. 7

Algorithm	Cardinality of BPS
Proposed Algorithm	**6**
Algorithm in [17]	7
Algorithm in [2]	6
Algorithm in [18]	11

TABLE V
RESULTS FOR IEEE 14-BUS SYSTEM SHOWN IN FIG. 8

Algorithm	Cardinality of BPS
Proposed Algorithm	**10**
Algorithm in [17]	13
Algorithm in [2]	13
Algorithm in [18]	18

such as the IEEE 30-bus system (Fig. 9), Indian Utility System (Fig. 10), and the western region 400-kV power grid (Fig. 11).

TABLE VI
RESULTS FOR IEEE-30 BUS SYSTEM SHOWN IN FIG. 9

Algorithm	Cardinality of BPS
Proposed Algorithm	**18**
Algorithm in [17]	19
Algorithm in [2]	20
Algorithm in [18]	43

TABLE VII
RESULTS FOR THE INDIAN UTILITY SYSTEM SHOWN IN FIG. 10

Algorithm	Cardinality of BPS
Proposed Algorithm	**19**
Algorithm in [17]	19
Algorithm in [2]	20
Algorithm in [18]	26

Hence, even though the techniques in [7] and [9] are elegant, they do not fare well in practice.

C. Results for Maximum-Independent Minimal BPS Computation

An efficient JAVA code was written for the computation of maximum-independent minimal BPS. The test systems considered are the same as those considered for computation of MBPS. For each system considered, it was found out that the cardinality of the maximum-independent minimal BPS was same as that of

TABLE VIII
NUMBER OF BUSES AND LINES IN EACH STATE IN THE 400-kV GRID IN WESTERN INDIA

Name of State	Buses	lines
Maharashtra	22	36
Madhya Pradesh	12	26
Gujarat	13	15
Chattisgarh	4	8
Goa	1	0

TABLE IX
BPS FOR WESTERN POWER GRID IN INDIA

Name of the state	Break Point Set
Chattisgarh	5
Gujarat	4
Maharashtra	25
Madhya Pradesh	23
Names of the Two states	**Break points on edges between them**
Madhya Pradesh, Gujarat	2
Madhya Pradesh, Chattisgarh	2
Madhya Pradesh, Maharashtra	3
Gujarat, Maharashtra	3
Maharashtra, Chattisgarh	3
Goa, Maharashtra	2
Total break points	**72**

TABLE X
LOWER BOUNDS ON THE NUMBER OF MINIMAL BPS

Name of System	Lower bound
IEEE 14 bus system in fig 8	135351
IEEE 30 bus system in fig 9	56144958
Indian Utility System in fig 10	449064
400 kV western grid in fig 11	5.15×10^{18}

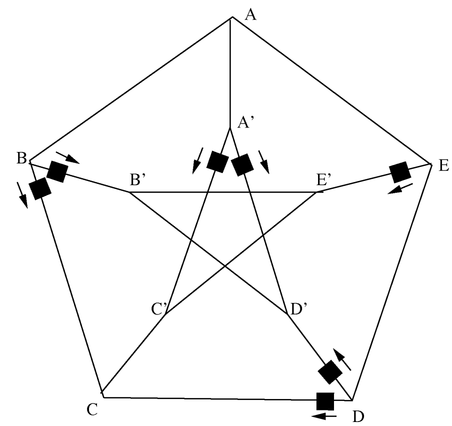

Fig. 12. Petersen graph and its MBPS.

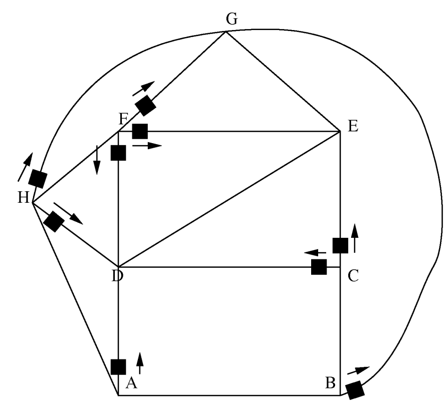

Fig. 13. Meshed system and its MBPS.

the MBPS. Therefore, the maximum-independent minimal BPS corresponds to one of the MBPSs. Also, for each test case, the solution to the relaxed LP converged directly to an integral solution.

VII. DISCUSSION

In designing solution approaches to NP-hard combinatorial optimization problems, one always has to accept that there would be a pathological instance or problem, where the nice features of the proposed approach would not hold. Since the optimal BPS computation problem also belongs to this class, we should anticipate exceptions where:

1) cost of the relaxed LP for MBPS computation will be strictly lower than the cardinality of the MBPS;
2) maximum-independent minimal BPS may not correspond to MBPS.

Our claim of the "goodness" of the proposed approach (formulation and methodology) is based on the fact that even after extensive tests, it was difficult to find such an exception. Nevertheless, we could find exceptions to the generic rule discussed before after very many evaluations.

Example: This example gives an exception of the 1st type. The Petersen graph is shown in Fig. 12. This graph has been extensively studied in graph theory [22]. On this topology, it was found that the cost of the relaxed LP for MBPS computation was lower than the cardinality of the MBPS. The relaxed LP has an optimum cost of 6 while the cardinality of the MBPS is 7. One possible MBPS has been indicated in the figure (blackened relays).

We could not find a system where a maximum-independent minimal BPS was not an MBPS. To support our claim of maximum-independent minimal BPS being a "better" BPS than MBPS, we provide an example where not every MBPS is a maximum-independent minimal BPS.

Example: Fig. 13 shows a case where the indicated MBPS is not a maximum-independent minimal BPS. A maximum-independent minimal BPS for the same system is indicated (blackened relays) in Fig. 14. In this case, it also turned out to be an MBPS. On the other hand, even after extensive analysis, we could not find a case where maximum-independent minimal BPS was not an MBPS. This issue requires further investigation.

Interestingly, the graph topology, which served as an example for the first exception, does not represent typical power system interconnections. The Petersen graph is a regular graph. Power system topologies in reality are not usually regular. This is one reason why we do not face such exceptions in practice.

VIII. CONCLUSION

In this paper, we have addressed two problems in optimal BPS computation.

1) Computation of minimum BPS.
2) Computation of maximum-independent minimal BPS.

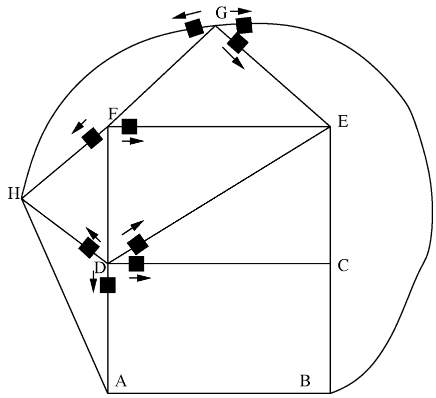

Fig. 14. Meshed system and its maximum-independent minimum BPS.

The problem of computing maximum-independent minimal BPS is posed for the first time in this paper. Further, we have proposed a generic ILP formulation for optimal BPS computation. Some of the salient features of this work are as follows.

- ILP formulation provides an efficient approach to the computation of optimal BPS. This is because more often than not, the relaxed LP formulation directly leads to an integral vertex.
- ILP approach can model phantom buses and relays with ease.
- Conceptually, it is more worthwhile to look for maximum-independent minimal BPS rather than MBPS. Fortunately, it is found that often one of the multiple MBPSs also corresponds to a maximum-independent minimal BPS.
- The paper provides an insightful and detailed investigation of the BPS problem from a graph-theoretic perspective.

Further, a decomposition approach was developed for large-scale systems. This led to a good approximation to the optimal BPS problem. Comparison with other methods in the literature demonstrates the efficacy of the proposed approach.

APPENDIX

Proposition 1.1: The number of minimal BPSs in any system is at least equal to the number of distinct trees in the corresponding undirected graph.

Proof: Consider any tree Γ in the undirected graph and let L denote the set of links in the graph corresponding to this tree. Let $\tilde{L}$ be the set of relays corresponding to these edges.

Claim: $\tilde{L}$ is a minimal BPS of the system

Proof: It is a BPS because if we remove all of the relays corresponding to $\tilde{L}$, then we are left with the tree Γ with all of its relays intact. However, this is a tree and, hence, it cannot have any loop. It is minimal because say if a particular relay (say, η) in the set $\tilde{L}$ is not removed, then the directed loop with one edge as η and the other edges on the tree remain intact and, hence, the set is not a BPS.

This shows that corresponding to every distinct tree, we have a distinct minimal BPS of the system. □

Using the Kirchoff Matrix–Tree theorem [23], we know that the number of distinct spanning trees in a graph is equal to the absolute value of any cofactor of the Laplacian of the graph. Hence, using I.1, we can conclude that the number of minimal BPSs in any graph is at least equal to the absolute value of any cofactor of the Laplacian of the graph. Except for the IEEE 14-bus system,[5] the lower bounds in Table X have been derived using this result.

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