CS281B/Stat241B (Spring 2008) Statistical Learning Theory **Lecture: 27**

Universal Portfolios With and Without Transaction Costs

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1 Introduction

We will be discussing the paper "Universal Portfolios With and Without Transaction Costs" by Avrim Blum and Adam Kalai [1]. In the paper, they show that an on-line investment algorithm proposed in [2] is competitive against the best constant rebalanced portfolio (CRP) determined in hindsight. They show that

$$\frac{\text{wealth of UNIVERSAL}}{\text{wealth of best CRP}} \geq \frac{1}{(n+1)^{m-1}} \tag{1}$$

and that in the presence of a commission $c \in [0, 1]$

$$\frac{\text{wealth of UNIVERSAL}}{\text{wealth of best CRP}} \geq \frac{1}{((1+c)n+1)^{m-1}}. \tag{2}$$

The importance of this result is that as $n \to \infty$

$$\frac{1}{n}\left[\log(\text{wealth of UNIVERSAL}) - \log(\text{wealth of best CRP})\right] \to 0. \tag{3}$$

Therefore, the daily performance of the universal portfolio performs as well as the best CRP.

2 Notation

Suppose that there are m stocks in a market, which we are interested in investing in over n days. On each day an investor will picks a way to distribute her money in the market. Let $\mathbf{b} \in \mathbb{R}^m$ be a vector indicating the distribution of an investor's wealth over all m stocks. $\mathbf{b}_i \geq 0$ and $\sum_i b_i = 1$. Let the initial value of the market be $\mathbf{x}_0 \in \mathbb{R}^m$, where $\mathbf{x}_{0j} \geq 0\ \forall j$. For every subsequent day let $\mathbf{x}_i$ denote the percent relative change of the market across day i. Thus, the value of stock j after the first day is $\mathbf{x}_{0j}\mathbf{x}_{1j}$, and more generally the value of a stock j on the i^{th} day is

$$x_{0j} \prod_{k=1}^{i} x_{kj}. \tag{4}$$

A constant rebalanced portfolio (CRP) is an investment strategy where $\mathbf{b}$ is fixed for each day. Thus, after each day, the proportion of total wealth allocated to stock j is $\mathbf{b}_j$. If during one day the price relatives are $\mathbf{x}$ and the wealth is allocated according to b, then the total increase in wealth is $\langle \mathbf{b}, \mathbf{x} \rangle = \sum \mathbf{b}_i \mathbf{x}_i$ and the total achieved wealth over n time steps is

$$S_n(\mathbf{b}, \mathbf{x}^n) = \prod_{i=1}^{n} \langle \mathbf{b}, \mathbf{x}_i \rangle. \tag{5}$$

2.1 Commission

The above formulation neglects any sort of commission costs. For simplicity, it is assumed that commission is a fixed percentage $c \in [0, 1]$ that is charged only for purhcases and not for sales. It can be assumed that an optimal trader will make transactions intelligently. For example, suppose that $c = 0.5$ and an investor has 100 dollar in stock A and 0 dollars in stock B and she would like to redistribute her wealth evenly among the two stocks. She could sell all of her stake in stock A, then buy 50 dollars of stock A and 50 dollars of stock B. Thus, she will be charged a total of 50 dollars. However, if she instead sells 50 dollars of stock A and purchases 50 dollars in stock B, she will only be charged 25 dollars. Please refer to the paper for a detailed analysis of how a trader should redistribute her wealth optimally.

For the purposes of the current paper, the following three assumptions can be made

1. The costs paid changing from distribute $\mathbf{b}_1$ to $\mathbf{b}_3$ is no more than the costs paid changing from $\mathbf{b}_1$ to $\mathbf{b}_2$ and then from $\mathbf{b}_2$ to $\mathbf{b}_3$.

2. The cost, per dollars, of changing from a distribution $\mathbf{b}$ to a distribution $(1 - \alpha)\mathbf{b} + \alpha\mathbf{b}'$ is no more than αc, because at most an α fraction of the money is being moved.

3. An investment strategy I which invests an initial fraction α of its money according to investment strategy I_1 and an initial $1 - \alpha$ of its money according to I_2, will achieve at least α times the wealth of I_1 plus $1 - \alpha$ times the wealth of I_2.

3 Analysis Without Commission

This section summarizes the analysis Cover's universal algorithm, without commission. Suppose we want a strategy that is competitive with respect to the best stock (ie. one that has the maximal worst-case ratio of your wealth to the best stock). A good strategy in this case is then to evenly divide your money among m stocks and let it sit. This deterministic strategy always has at least $\frac{1}{m}$ times as much money as the best stock, and achieves the expected wealth of a random strategy.

The same analogy extends to competing with the best CRP. Cover's universal portfolio algorithm splits the money evenly among all CRPs and let it sit in these strategies (there is no money transfer between the strategies). The formal definition is as follows:

Definition. (UNIVERSAL) The universal portfolio algorithm at time i is specified by

$$\hat{b_i} = \frac{\int_\beta \mathbf{b} S_{i-1}(\mathbf{b}, \mathbf{x}^{\mathbf{i-1}}) d\mu(\mathbf{b})}{\int_\beta S_{i-1}(\mathbf{b}, \mathbf{x}^{\mathbf{i-1}}) d\mu(\mathbf{b})}, \text{ i = 1,2,...}$$

with μ being the uniform distribution over portfolios $\mathbf{b}$.

Cover notes in [2] that

$$\text{wealth of UNIVERSAL} = E_{\mathbf{b} \in \beta} [\text{ wealth of } CRP_b \text{ }].$$

This is analogous to the previous case while competing with the best stock. The corresponding worst-case ratio then satisfies the following.

Theorem 3.1. (As stated in [2]),

$$
\begin{aligned}
\frac{\text{wealth of UNIVERSAL}}{\text{wealth of best CRP}} &\geq \binom{n+m-1}{m-1}^{-1} \\
&\geq \frac{1}{(n+1)^{m-1}}, \\
&\quad \text{for all markets with m stocks and n periods}
\end{aligned}
$$

Proof. (Summary of proof. Check paper for details) The idea is that portfolios that are "near" each other perform similarly, and there is a large fraction of portfolios "near" the optimal one.

Suppose in hindsight $\mathbf{b}^*$ is the optimal CRP. Let $\mathbf{b} = (1-\alpha)\mathbf{b}^* + \alpha\mathbf{z}$, for some $\mathbf{z} \in \beta$. (In other words, $\mathbf{b}$ is close to $\mathbf{b}^*$.) For a single period, gain of $CRP_b \geq (1-\alpha)$ (gain of CRP_{b^*}). Over n periods,

$$\text{wealth of CRP}_b \geq (1-\alpha)^n(\text{wealth of CRP}_{b^*}). \tag{6}$$

Then, we get

$$
\begin{aligned}
\frac{\text{wealth of UNIVERSAL}}{\text{wealth of best CRP}} &\geq E_{\mathbf{b}\in\beta}[(1-\alpha)^n] \\
&= \int_0^1 Prob_{\mathbf{b}\in\beta}[(1-\alpha)^n \geq x]dx \\
&= \int_0^1 (1-x^{1/n})^{m-1}dx \\
&= n\int_0^1 y^{n-1}(1-y)^{m-1}dy \\
&= ... \\
&= n(\frac{(m-1)!(n-1)!}{(n+m-2)!}) \\
&= \frac{1}{\binom{n+m-1}{m-1}}
\end{aligned}
$$

□

3.1 Randomized Approximation

An exact implementation requires $O(n^{m-1})$ in space and time complexity. It is possible to use a randomized approximation. First choose N portfolios at random, invest $\frac{1}{N}$ of the money in each, and let it sit within each CRP. If the best CRP achieves a wealth R times UNIVERSAL's wealth, then Chebyshev's inequality guarantees that using N = $\frac{R-1}{\epsilon^2\delta}$ random portfolios, the approximation achieves a wealth at least 1-ϵ times that of UNIVERSAL, with probability at least 1-δ. Although R can potentially grow like n^{m-1}, experiments on stock market data [2] all have R ¡ 2.

The same analysis and implementation can be extended to the $Dirichlet(\frac{1}{2}, ..., \frac{1}{2})$ prior, instead of the uniform prior.

4 Analysis with Commission

In this section, commission is introduced and a slight modification to UNIVERSAL, $UNIVERSAL_c$, is considered. At the start of the i^{th} period, UNIVERSAL computes a weighted average of the CRPs. The weight of a particular CRP_b is proportional to the weight it has accumulated during the first *i-1* periods, $S_{i-1}(\mathbf{b}, \mathbf{x}^{i-1})$. Similarly, $UNIVERSAL_c$ computes a weighted average of the CRPs. The weight of a particular CRP_b is now proportional to the wealth it has accumulated *including commission costs*,$S^c_{i-1}(\mathbf{b}, \mathbf{x}^{i-1})$. We can formally define $UNIVERSAL_c$ in a similar fashion:

Definition. ($UNIVERSAL_c$) The universal portfolio algorithm at time i is specified by

$$\hat{b_i}^c = \frac{\int_\beta \mathbf{b} S^c_{i-1}(\mathbf{b},\mathbf{x^{i-1}})d\mu(\mathbf{b})}{\int_\beta S^c_{i-1}(\mathbf{b},\mathbf{x^{i-1}})d\mu(\mathbf{b})}, \text{ i = 1,2,...}$$

with μ being the uniform distribution over portfolios $\mathbf{b}$.

Like the case without commission, $UNIVERSAL_c$ achieves the expected wealth of a random CRP. We can derive a similar lower bound on the worst-case ratio between the wealth of the algorithm and that of the best CRP.

Theorem 4.1. In the presence of commission $o \le c \le 1$,

$$\begin{aligned} \frac{\text{wealth of UNIVERSAL}_c}{\text{wealth of best CRP}} &\ge \binom{(1+c)n+m-1}{m-1}^{-1} \\ &\ge \frac{1}{((1+c)n+1)^{m-1}}, \\ &\quad \text{for all markets with m stocks and n periods} \end{aligned}$$

Proof. Based on the three properties given in Section 2.1, if $b_j \ge (1-\alpha)b^*_j$, then

$$\frac{\text{single-period profit of CRP}_b}{\text{single-period profit of CRP}_{b^*}} \ge (1-\alpha)(1-c\alpha). \tag{7}$$

Consult the paper for details.

Over n periods, this gives

$$\text{wealth of CRP}_b \ge (1-\alpha)^{(1+c)n}(\text{wealth of CRP}_{b^*}). \tag{8}$$

The previous proof then follows, and we can replace n by (1+c)n in the final guarantee. □

The algorithm can be implemented with the same randomized approximation as explained previously.

5 Semi-constant-rebalanced Portfolios

A semi-constant-rebalanced portfolio (SCRP) was proposed as a good strategy in the presence of transaction costs. The portfolio can be rebalanced on any subset of the periods rather than after each time period. This strategy could be beneficial if cost of rebalancing outweighs the benefits. In the paper the authors show that no strategy can guarantee the same expected exponential growth rate as the best SCRP selected in hindsight.

Consider the set of all market sequences of length n consiting of two stocks. In each period suppose that either one of the stocks crashes by a fraction $1 >> \epsilon > 0$ while the other stock remains constant. Therefore, the price relatives in each day are either $(1, \epsilon)$ or $(\epsilon, 1)$. Thus, if $\mathcal{K}$ denotes the sequences of markets, then

$$\mathcal{K} = \{\mathbf{x}^n : \mathbf{x}_i = (1, \epsilon) \text{ or } \mathbf{x}_i = (\epsilon, 1), \forall i \leq n\}. \tag{9}$$

Suppose that an element of $\mathcal{K}$ is selected uniformly over the entire set. Then the following theorem holds.

Theorem 5.1. Any non-anticipiating investment strategy (a strategy which has no knowledge of the future) will achieve an expected wealth of $\left(\frac{1+\epsilon}{2}\right)^n$.

Proof. For each time period i let

$$f_i(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}) = (p_i, 1 - p_i) \tag{10}$$

for $p_i \in [0, 1]$ be the distribution of the investor's wealth over the two stocks for time period i. Then over n time periods

$$\begin{aligned}
E[\prod_{i=1}^{n} \langle f_i, \mathbf{x}_i \rangle] &= E[E[\prod_{i=1}^{n} \langle f_i, \mathbf{x}_i \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}]] && (11)\\
&= E[E[\langle f_n, \mathbf{x}_n \rangle \prod_{i=1}^{n-1} \langle f_i, \mathbf{x}_i \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}]] && (12)\\
&= E[E[\langle f_n, \mathbf{x}_n \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}] E[\prod_{i=1}^{n-1} \langle f_i, \mathbf{x}_i \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}]] && (13)\\
&= E[\left(\frac{1+\epsilon}{2}\right) E[\prod_{i=1}^{n-1} \langle f_i, \mathbf{x}_i \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}]] && (14)\\
&= \left(\frac{1+\epsilon}{2}\right) E[E[\prod_{i=1}^{n-1} \langle f_i, \mathbf{x}_i \rangle | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}]] && (15)\\
&= \left(\frac{1+\epsilon}{2}\right) E[\prod_{i=1}^{n-1} \langle f_i, \mathbf{x}_i \rangle] && (16)\\
&= \left(\frac{1+\epsilon}{2}\right)^n, && (17)
\end{aligned}$$

where the equalities in (13) and (14) follow because conditioned on $\mathbf{x}^{n-1} = \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$, f_n is a fixed vector and $\mathbf{x}_n$ is independent of $\mathbf{x}^{n-1}$. The final equality follows by induction. □

Thus, any strategy with no knowledge of the future will expect to achieve $(\frac{1+\epsilon}{2})^n$ wealth. However, an SCRP can do much better.

Let P be an SCRP strategy such that the portfolio is rebalanced such that the wealth is evenly spread between both stocks. Therefore, the wealth is evenly divided between both stocks initially. The investor will then only rebalance when the market is about to switch ($\mathbf{x}_i \neq \mathbf{x}_{i-1}$). Suppose that the market switches k times. Then, the total wealth obtained with this strategy is greater than or equal to $\frac{1}{2^{k+1}}$. Thus, the expected wealth of P taken over the uniform distribution on $\mathcal{K}$ is at least

$$\frac{1}{2^n} \sum_{k=0}^{n-1} 2 \binom{n-1}{k} \frac{1}{2^{k+1}} = \frac{1}{2^n} \left(1 + \frac{1}{2}\right)^{n-1}. \tag{18}$$

Therefore, the SCRP strategy using hindsight is at least $\left(\frac{1}{1+\epsilon}\right)^n \left(\frac{3}{2}\right)^{n-1}$ times greater than the expected performance of any non-anticipating strategy. Thus, establishing that for a small enough $\epsilon > 0$, an SCRP with hindsight can exponentially outperform any non-anticipating investment strategy.

References

[1] Blum and Kalai. Universal portfolios with and without transaction costs. In *COLT: Proceedings of the Workshop on Computational Learning Theory, Morgan Kaufmann Publishers*, 1997.

[2] T.M. Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, January 1991.