PRICE OF ANARCHY

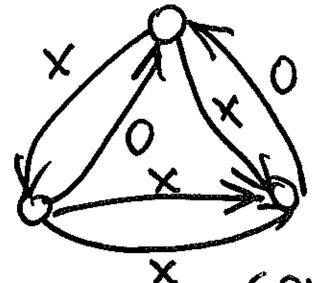
$$PoA = \frac{\max_{a \text{ equilibrium}} \sum c_i(a)}{\min_a \underbrace{\sum c_i(a)}_{\text{objective = sum of agents' objectives}}}$$

Assume minimization problem

Q: • what equilibrium? (Nash, later, correlated, mag
• why unweighted sum (context often supports it)
• why didn't economists study it?

(Atomic) Congestion Game

e.g.



4 flows 1 : a → b
2 : a → c
3 : b → c
4 : c → b

congestion fcn x : (1,2,3,4)

pure Nash eq. exists: potential fcn

$$\Phi(a) = \sum_e \sum_{j=1}^{f_e(a)} c_e(j)$$

choice of paths resulting in flow $f(a)$

Potential game

<u>Note</u>: $\Phi(a) - \Phi(a') = c_i(a) - c_i(a')$

a': flow after defection of i ⇒ pure Nash eq always exists!

Example has two pure Nash eq.

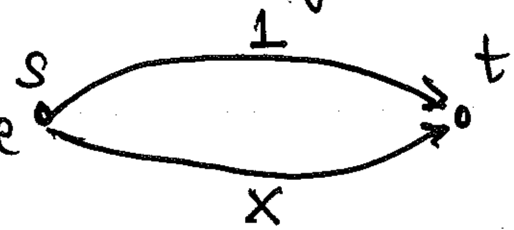
Single hop → $\sum_i c_i(a) = 4$

Double hop → $= 10$
(also a Nash eq!)

PoA $= 2.5$. This is worst possible! (see below)

<u>Note</u>: It does make sense to want to optimize <u>unweighted</u> sum of time wanted, energy, etc.

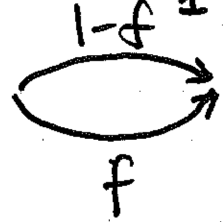
Non-atomic flow: A large population of agents of total volume 1 want to go from s to t
continuum of agents/drivers

Pigou's example


x: total cost x^2
per-flow cost x

Only Nash eq: 

cost 1

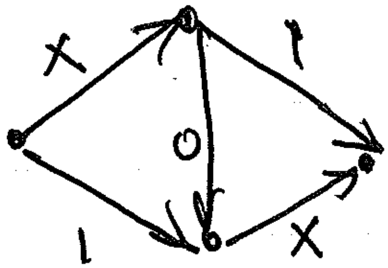
optimum? 

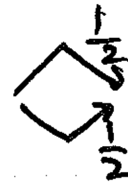
min $1-f+f^2 \Rightarrow f=\frac{1}{2}$

cost $= \frac{3}{4}$

PoA $= \frac{3}{4}$. Worst possible!

Braess's Paradox



opt: 

1.5

N.eq: 

2

p.o.A $= \frac{4}{3}$ coincidence?

What do we know about optimum and Nash eq. flows in such networks?

various kinds of costs	$c_e(x)$	(assume increasing) cost per unit of flow
	$C_e(x) = x \cdot c_e(x)$	cost (convex)
	$c^*_e(x) = C'(x) = c_e(x) + x \cdot c_e'(x)$	marginal cost

<u>Thm</u> A Nash eq. flow is optimum flow in the c^*_e that it creates. That is, all used paths have equal c^*_e length, all unused ones worse.

<u>In Pigou, Braess:</u> x^* $\overset{\text{Nash}}{(\text{eq})}$ equalizes path lengths.

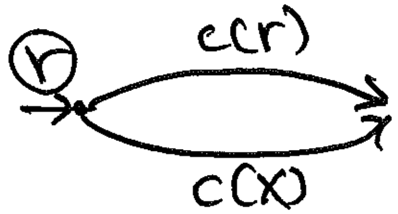
<u>Thm</u> Equilibrium flow iff global minimum of $\sum_e \int_0^{f_e(x)} c_e(x)\,dx = \Phi(x)$

<u>Proof</u>: $x \cdot c_e(x) \Leftrightarrow \int_0^{f_e(x)} c_e(x)\,dx$

<u>Cor</u>: Equilibrium exists, its cost is unique (because cost is convex combo.)

Notice immediately: for some fixed class of congestion fcns

$$\text{PoA} \geq B = \max_{c \in \mathcal{C};\, x, r \geq 0} \frac{c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

<u>Proof</u>: 

(gen'd Pigou)

<u>Note</u>: for $\mathcal{C} = \{ax+b : a, b \geq 0\}$, $B = \frac{4}{3}$ (b?)

$$B \geq \frac{f^* c(f^*)}{\hat{f} c(\hat{f}) + (f^* - \hat{f})\, c(f^*)} \qquad \left[\begin{array}{l} \text{set } r = f^* = \text{eq} \\ \hat{f} = \text{opt} \end{array}\right]$$

or

$$\hat{f} c(\hat{f}) + \underbrace{c(f^*)(f^* - \hat{f})}_{\substack{\geq 0 \\ \text{because } f^* \text{ is eq.}}} \geq \frac{1}{B} c(f^*) f^*$$

<u>QED</u>

$\therefore B = \text{PoA}!$

<u>Amazing fact</u>: PoA is determined by the class $\mathcal{C}$, not by the network. Always attained at the trivial network!

Atomic congestion games?

Lemma Suppose that $\exists \lambda>0, \mu<1$ $\forall s, s', i$

$$c_i(s_i', s_{-i}) \le \lambda \cdot c_i(s') + \mu \cdot c_i(s)$$

then $PoA \le \lambda / (1-\mu)$

Pf $cost(s^*) = \sum_i c_i(s^*) \le \sum_i c_i(\hat{s}_i, s^*_{-i})$

$\uparrow$ eq $\Rightarrow$ $\uparrow$ $\quad\uparrow$ opt

$\le \lambda \cdot cost(\hat{s}) + \mu \cdot cost(s^*)$. QED.

To apply to atomic congestion:

$$\sum c_i(x_i', x_{-i}) \le \sum_e (a_i(x_e+1)+b_e)\, x_e$$

now: $\boxed{\forall a,b,y,z: \ ay(z+1)+by \le \frac{5}{3}(ay+b)y + \frac{1}{3}(az+b)z}$

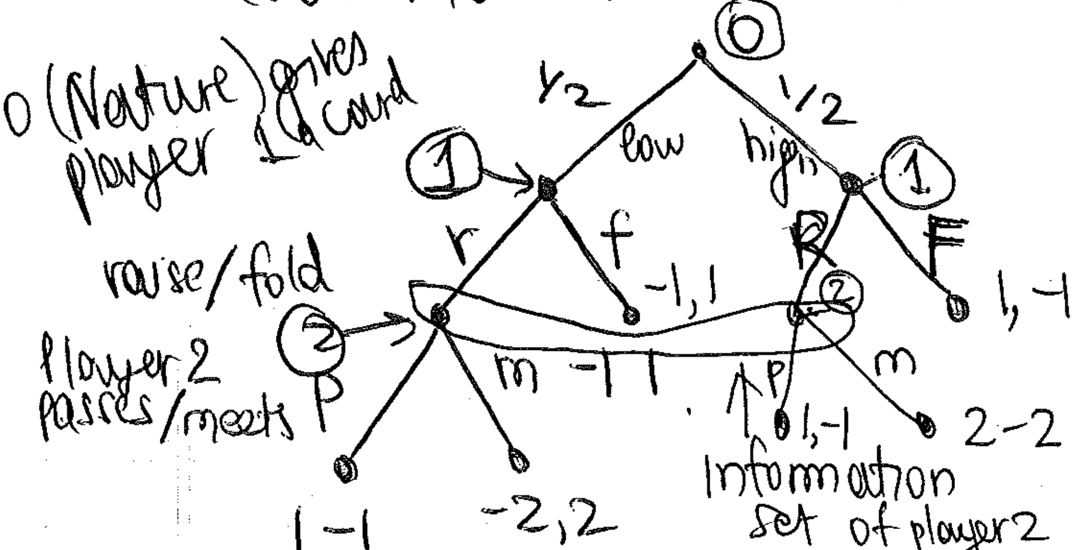
$$\le \frac{5}{3}\sum_e (a_e x_e' + b_e)\, x_e' + \frac{1}{3}\sum_e (a_e x_e + b_e)\, x_e$$

$\Rightarrow PoA \le \dfrac{5/3}{1-1/3} = \dfrac{5}{2}$ (matches triangle lower bound)

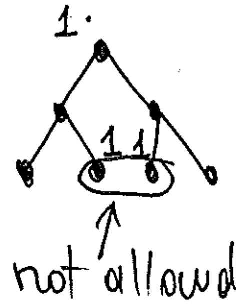
Also: holds not just for Nash eq, but also for no-regret play (boosting/experts) which also means for correlated equilibria. (apply the inequality at each time t...)

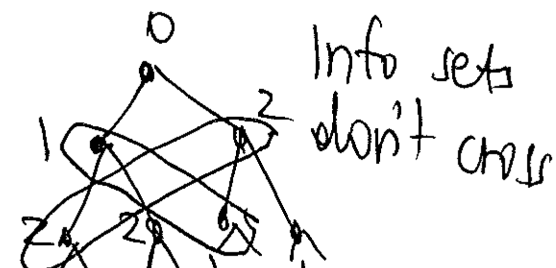
EXTENSIVE FORM GAMES

(We must include chess in our games....)



Important restrictions on the trees: & Info sets: | Perfect recall





Can be reduced to the table form. But there are issues the reduction is exponential in general. And Nash eq. is no good here!

	m	p
fF	0 0	0 0
rF	$-\frac{1}{2}$ $\frac{1}{2}$	1 -1
fR	$\frac{1}{2}$ $-\frac{1}{2}$	0 0
rR	0 0	1 -1

What is a mixed strategy?
- mixing all rows/col (exponential)
- behavioral (mix at each info set)
- sequence trick (von Stengel, Kohler)

Q: Can you see why we can go the other way?
table form → extensive form

Sequence trick [Kohler + von Stengel 93]

Notation H_i info sets for player i
$h \in H_i$ has A_h (available actions)

next: $H_i \times A_h \to H_i \cup \{\text{leaves}\}$

this↑ is the "tree", valid because of perfect recall (depending on i)

Assign probabilities to the h's (prob of getting there)

For the topmost $h \in H_i$, $x_i(h) = \text{const}$ (depends on player 0)

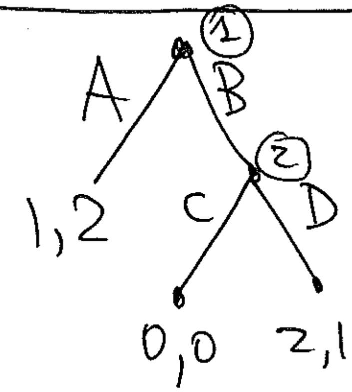
$$x_i(h) = \sum_{a \in A_h} x_i(\text{next}(h,a)) \quad \text{[linear!!]}$$

We can now write the expected payoff of a mixed action profile:

$$E[u_i] = \sum_{\ell \in \text{Leaves}} u_i(\ell) \prod_{i=1}^{n} x_i(\ell)$$

Now we can solve by LP zero-sum two-person games.... (like poker!)

Nash is no good!



	C	D
A	1,2	1,2
B	0,0	2,1

Two Nash eq. Upper-left is "empty threat"!

Solution: Subgame Perfect Nash eq.

Subgame: ~~[illegible]~~ subtree closed under info sets

SpNeq: A Nash eq. that is also a Nash eq when restricted to any subgame

No info sets: Neq. can be computed easily, and is subgame perfect; by DP from the leaves up (backwards induction or Zermelo's algorithm).

Info sets: Same idea. Compute Nash eq. of subtrees bottom up. Replace subtrees by payoffs of Nash eq. on a leaf. Continue.

Another solution Trembling hand Nash eq.
A Nash eq x such that there is a sequence (x_j) of ε-fully mixed "Nash eq" (all strategies best responses under the constraints $x_i \geq \varepsilon^i > 0$), such that $x_j \to x$.

Algorithm? Conjecture: Find ε-fully mixed Nash eq with ε_i 2^{-n^2}. Round down/up

Third fix Sequential Equilibrium | Game theorists consider it the "final" solution concept

For every info set a set of "beliefs" β_h and a (behavioral) mixed strategy x_h. β_h is a distribution over all paths to h ("how did I get here?"). x_h shoud be best response assuming β.

Furthermore: Sequence of (x^j, β^j) such that $x^j \to x$, $\beta^j \to \beta$, x^j fully mixed, β^j derived from x^j via Bayes rule...

Algorithm ? Open

Bayesian Games

player i: actions A_i
types T_i
utility $u_i: \prod_j A_j \times \prod_j T_j \to \mathbb{R}$
prior: $\pi_i \in \Delta \prod_j T_j$

Example: Vickrey auction!

Reduce to table form: action set $B_i : T_i \to A_i$, payoffs

$u_i(b_1 .. b_n) = E_{\pi_i}[u_i(b_j(t_j), t_j, j = 1..n)]$

<u>Theorem</u> In Vickrey, b_i = the identity function is dominant.

<u>Harsanyi's Theorem</u> (justification of mixed Nash eq)

For almost all games G there is a sequence of Bayesian games $\hat{G}_j$ such that $\hat{G}_j \to G$ and the equilibria of $\hat{G}_j$ converge to the equilibria of G

payoffs of $\hat{G}$: $u_{ij} + \varepsilon_{ij}^{k} \leftarrow$ (perturbed)

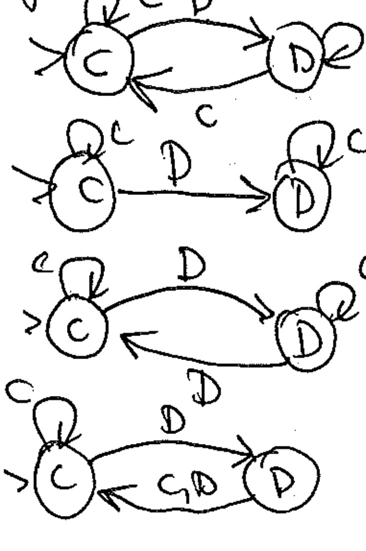
Another special Case: Repeated Games

Repeat Prisoner's Dilemma n times!

<u>Fact</u>: Defect n times is the only Nash eq (backwards induction)

<u>One Path taken in the 1980's–1990's</u>: Assume players are limited. e.g., automata with few states!

<u>2 states</u>: 4 possible strategies undominated



"tit-for-tat"

"U.S. foreign policy"

"switch on D"

"punish once"

$< n$ states: collaborate possible, not enough states to count to n & start backwards induction...

$\geq 2^n$ states D-D (enough states to run the DP algorithm- too smart for their own good)

poly(n) complicated collaboration... [Pap. + Kannatakis, STOC 94]

Infinite repetition $\hat{A}_i = \left(\bigcup_{t=0}^{\infty} A^t \to A_i \right)$

look at your history of all players, decide what to do next.

defines $a(t)$

$$u_i(t) = \sum_{t=1}^{\infty} u_i(a(t)) \cdot (1-\delta)^t$$

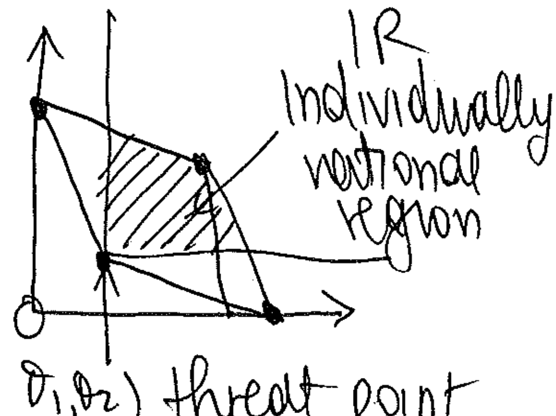
discount factor.

Important Insight: The Folk Theorem (Shapley, Aumann, Maskin ~1970's-80's)

PD

4 4	5 0
0 5	1 1

$\Rightarrow$



(v_1, v_2) threat point

$$v_i = \min_{a_{-i}} \max_{a_i} u_i(a) \quad \text{in one-shot game}$$

(R)

Thm Any rational ($= p/q$) point in $IR > \vartheta$ can be realized by repeated play (including C-C in PD(ϑ)).

Proof: Realize it by periodic play. If anybody defects, the rest will punish him!

(Objection: not subgame perfect. Subgame perfect version: punish n times, if anybody defects from *that* punish *them* n^2 times, etc.)

For two players can be realized.
(Because $\vartheta_1 = \min\max(-B, B)$)

But > 2 players? [turns out that computing ϑ_i is NP-hard...]

Thm Finding an eq. in repeated game (i.e., any point in IR...) is as hard as in ~~the~~ one-shot games

Proof. Take k-player game G. Create $(k+1)$-player game $\hat{G}$ with player $k+1$ (the "kibitzer"). $A_{k+1} = \{(j,a) : a \in A_j\}$

$$u_i(a, (j, b)) = \begin{cases} 0 & \text{if } i \neq k+1, j \\ u_j(b, a_{-i}) - u_j(a) & \text{if } i = k+1 \\ u_j(a) - u_j(b, a_{-i}) & \text{if } i = j \end{cases}$$

j: player, $\leq k$; $b \in A_j$

At equilibrium, k players will play a Nash eq. of the one-shot game...

Coalitional Games

$[n]$ players. $v : 2^{[n]} \to \mathbb{R}^+$

$v[S]$: the worth of coalition S.

<u>assumptions</u> $v(\emptyset) = 0$, $\sum_{S \in \Pi} v(S) \leq v([n])$ ← partition of $[n]$

In games, atom of behavior = action of player
Here, what a <u>coalition</u> does

Basic question: How do the n players split $v([n])$?

One solution concept: the <u>core</u>
imputation: x_i with $\sum x_i = v([n])$
or payoff profile
feasible

S-feasible: $\sum x_i = v(S)$

x is in the core iff there is no "defection", coalition S and S-feasible y s.t. $y_i > x_i$ $\forall i \in S$.

Ex 1 $v([3]) = 1$, $v(S) = \alpha$ if $|S| = 2$, $v(S) = 0$ if $|S| \leq 1$.

Ex 2 How should n cities split road maint'ce costs?

Ex 3 Voting $v(S) = \begin{cases} 0 \\ 1 \end{cases}$ either core $= \emptyset$ or there is a veto player!

Ex 4 Game G, n players. Define $v(S) = \min_{[n]-S} \max_{S} \sum_{i \in S} u_i$

the bottom-line total utility players in S can guarantee for themselves

Ex 5 A trip to n destinations. How do you apportion the airfare to the n hosts?

Shapley value

$$S_i = \frac{1}{n!} \sum_{\pi \in S_n} v(\{j : \pi(j) \leq \pi(i)\}) - v(\{j : \pi(j) < \pi(i)\})$$

Thm. If the core is nonempty, $S \in$ core

Thm S is the unique value satisfying the balanced contributions property:

$$S_i([n]) - S_i([n] - \{j\}) = S_j([n]) - S_j([n] - \{i\})$$

Also: Symmetry, additivity, dummy player