CS 70 Fall 2003 — Discrete Mathematics for CS — Wagner — HW 4

Due Thursday September 25

1. (5 pts.) Any questions?
Is there anything you'd like to see explained better in lecture or discussion sections?

2. (25 pts.) Stable marriage
Consider a set of four boys (a, b, c, d) and four girls (1, 2, 3, 4) with the preferences shown in Figure 1.

(a) Run on this instance the traditional marriage algorithm. Show each stage of the algorithm, and give the resulting matching, expressed as a set of boy-girl pairs. You can do this by hand, or you can write a simple program to do it for you.

For instance, you might show this by drawing a picture for each day, with dotted lines from each boy to the girl he visits that day and with solid lines between couples who will be "engaged" at the end of the day.

(b) The matching you found above is boy-optimal. Find now a girl-optimal stable matching. (It requires running a modified algorithm.) Compare the two matchings.

boy	preferences	girl	preferences
a	1>2>3>4	1	d>b>c>a
b	2>1>4>3	2	a>d>b>c
c	1>3>2>4	3	a>b>c>d
d	2>1>3>4	4	d>c>a>b

Figure 1: Preferences for the stable marriage problem. ($1 > 2$ means that 1 is preferred to 2.)

3. (30 pts.) Cake-cutting
Consider this protocol for three-party cake cutting:

1. Alice cuts the cake into three equal chunks (equal by her measure).
2. Bob cuts each of these three chunks in half (so that both halves of each chunk are equal by his measure).
3. Among these 6 pieces, Carol chooses the two best pieces (by her measure).
4. Among the 4 remaining pieces, Alice chooses the best two (by her measure).
5. Bob gets the last 2 pieces.

Justify each of your answers below briefly.

(a) Is this protocol fair for Alice? for Bob? for Carol?

(b) Is this protocol envy-free for Alice? for Bob? for Carol?

Recall that a protocol is envy-free for X if it has the following property: if X follows the protocol, then X gets at least as much (by her measure) as anyone else gets, no matter how the other parties behave.

4. (20 pts.) You be the grader, again

Let the sequence $a_0, a_1, a_2, \dots$ be defined by the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n \geq 2 \qquad \text{and} \qquad a_0 = 1, a_1 = 2.$$

A visiting professor from that private school across the Bay proposes the following argument:

Theorem 0.1: *$a_n \leq n+2$ for all $n \geq 0$.*

Proof: We use strong induction on n. The base cases $n = 0$ and $n = 1$ hold, since $a_0 = 1 \leq 0+2$ and $a_1 = 2 \leq 1+2$. For the induction step, fix some $n \geq 2$, and suppose $a_i \leq i+2$ for each $i = 0, 1, \dots, n-1$. In this case, we have

$$a_n = 2a_{n-1} - a_{n-2} \leq 2((n-1)+2) - ((n-2)+2) \leq 2n+2-n \leq n+2,$$

which shows that $a_n \leq n+2$ holds for all $n \geq 0$. □

(a) Critique the above proof.

(b) Give a better proof of the theorem.

5. (20 pts.) Paradox

Consider the following result, first proved by Euclid many centuries ago.

Theorem 0.2: *There exist infinitely many primes.*

Proof: Assume to the contrary that there exist finitely many primes. Let these primes (in increasing order) be $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\dots$, p_k. Let $q_k = p_1 p_2 p_3 \cdots p_k + 1$. Note that q_k is a new number not in the list of primes $p_1, \dots, p_k$. At the same time, it is not divisible by p_i for any i, since q_k is one more than a multiple of p_i. This means that q_k is a new prime different from $p_1, \dots, p_k$, which is a contradiction. This completes the proof. □

(a) Is the above proof valid? (You do not need to justify your answer.)

(b) Let $p_1, \dots, p_k$ represent the first k primes. Are we guaranteed that $p_1 p_2 p_3 \cdots p_k + 1$ is always prime for all $k \geq 1$? Justify your answer.

6. (0+10 pts.) Bonus: Can it really be this easy?

Optional bonus problem: Here is a cheesy algorithm for the stable marriage problem:

1. Start with a pairing where the i-th boy is engaged to the i-th girl, for all i.
2. While there exists a rogue couple (b, g^*) in the current pairing, do:
3. Suppose b is currently engaged to g and b^* is currently engaged to g^*.
4. Have b, g, b^*, g^* swap partners. After the swap, b will be engaged to g^*, and b^* to g.
5. Marry off all the engaged couples.

Obviously, if the cheesy algorithm terminates, the result is a stable pairing.

Prove or disprove: The cheesy algorithm is always guaranteed to terminate, *no matter what the preferences are and no matter how any ambiguity in line 2's choice of a rogue couple is resolved during the execution of this algorithm.*