CS 70 Fall 2003 — Discrete Mathematics for CS, Wagner — HW 9

Due Thursday November 13

1. (5 pts.) Any questions?
Is there anything you'd like to see explained better in lecture or discussion sections?

2. (15 pts.) Counting
How many non-negative integer solutions $(x_1, \ldots, x_7)$ are there to the following equation?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 2003$$
$$x_1 \geq 0, \ldots, x_7 \geq 0, \qquad x_1, \ldots, x_7 \in \mathbf{Z}$$

Order matters. For instance, $(1, 2002, 0, 0, 0, 0, 0)$ counts as a different solution than $(2002, 1, 0, 0, 0, 0, 0)$.

3. (30 pts.) A paradox in conditional probability?
Here is some on-time arrival data for two airlines, A and B, into the airports of Los Angeles and Chicago. (Predictably, both airlines perform better in LA, which is subject to less flight congestion and less bad weather.)

	Airline A		Airline B	
	# flights	# on time	#flights	# on time
Los Angeles	600	534	200	188
Chicago	250	176	900	685

(a) Which of the two airlines has a better chance of arriving on time into Los Angeles? What about Chicago?

(b) Which of the two airlines has a better chance of arriving on time overall?

(c) Do the results of parts (a) and (b) surprise you? Explain the apparent paradox, and interpret it in terms of conditional probabilities.

4. (30 pts.) Happy families

(a) Consider a collection of families, each of which has exactly two children. Each of the four possible combinations of boys and girls, bg, gb, bb, gg, occurs with the same frequency. A family is chosen uniformly at random, and we are told that it contains at least one girl. What is the (conditional) probability that the other child is a girl? In other words, what is the (conditional) probability that both children are girls, given that the family we chose contains at least one girl? Justify your answer with a precise calculation.

(b) On the same probability space as in part (a), let A be the event that the chosen family has children of both sexes, and B the event that the family has at least one girl. Are the events A and B independent? Justify your answer carefully.

(c) Consider now the probability space of families with *three* children, with each of the eight possible combinations of boys and girls equally likely. Define the events A and B as in part (b). Are these events independent? Again, justify your answer carefully.

5. (20 pts.) De Méré's 2nd problem

Last homework, we saw the first problem that M. le chevalier de Méré posed to Pascal and Fermat in the 17th century. Here is the second problem, or at least, my own personal interpretation of the essence of his problem.

Two players, Alice and Bob, each stake 32 pistoles on a three-point, winner-take-all game of chance. When Alice has 2 points and Bob has 1 point, the game is interrupted and cannot continue. How should the stakes of 64 pistoles be fairly distributed?

You should assume that the game is played in rounds, at each round of the game one of the two players gains a point (and the other gains none), and that Alice and Bob are evenly matched, so that in each round Alice and Bob each have a 50% chance of winning the round. If the game were played to the end, the first to win 3 points would take home the whole 64 pistoles. Of course, we cannot continue the game, so we have to come up with an equitable division into partial shares.

Fermat's proposal is lost to history, but here is a desiderata that might be suggested today: Alice's share should be proportional to the maximum that Alice should be willing to pay to continue to the game from the point where it was interrupted. Economics tells us[1] that the maximum Alice should be willing to pay is exactly the conditional expected value of her winnings (specifically, her winnings if the game were continued to the end from this point). The same goes for Bob.

Calculate a fair way to distribute the 64 pistoles using this notion of fairness. How many pistoles does Alice receive? Bob?

[1] This is a little bit of a simplification. I'm assuming here that Alice's utility function is the identity function. That assumption might or might not be perfect in practice, but it's probably a reasonable first approximation for low-stakes games.