Everything You Always Wanted To Know about Game Theory*

*but were afraid to ask

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What is "Game Theory"?

Combinatorial / Computational / Economic

Combinatorial

- Sprague and Grundy's 1939 *Mathematics and Games*
- Board (table) games
- Nim, Domineering
- Complete info, alternating moves
- Goal: Last move

Computational

- R. Bell and M. Cornelius' 1988 *Board Games around the World*
- Board (table) games
- Tic-Tac-Toe, Chess
- Complete info, alternating moves
- Goal: Varies

Economic

- von Neumann and Morgenstern's 1944 *Theory of Games and Economic Behavior*
- Matrix games
- Prisoner's dilemma
- Incomplete info, simultaneous moves
- Goal: Maximize payoff

Know Your Audience…

- How many have used games pedagogically?
- What is your own comfort level with GT? (hands down = none, one hand = ok; two hands = you could be teaching this session)
  - ◊ Combinatorial (Berlekamp-ish)
  - ◊ Computational (AI, Brute-force solving)
  - ◊ Economic (Prisoner's dilemma, matrix games)

EYAWTKAGT*bwata
Here's our schedule:

("GT" = "Game Theory")

- Dan: Overview, Combinatorial GT basics
- David: Combinatorial GT examples
- Dan: Computational GT
- Peter: Economic GT & Two-person games
- Dan: Summary & Where to go from here

(All of GT in 75 min? Right!)

Why are games useful pedagogical tools?

- Vast resource of problems
  - ◊ Easy to state
  - ◊ Colorful, rich
  - ◊ Use in lecture or for projects
  - ◊ They can USE their projects when they're done
  - ◊ Project Reuse -- just change the games every year!
  - ◊ Algorithms, User Interfaces, Artificial Intelligence, Software Engineering

"Every game ever invented by mankind, is a way of making things hard for the fun of it!"

– John Ciardi

What is a combinatorial game?

- Two players (Left & Right) *alternating turns*
- No chance, such as dice or shuffled cards
- Both players have perfect information
  - ◊ No hidden information, as in Stratego & Magic
- The game is finite – it must eventually end
- There are no draws or ties
- Normal Play: Last to move wins!

Combinatorial Game Theory
The Big Picture

- Whose turn is not part of the game
- SUMS of games
  - ◊ You play games $G_1 + G_2 + G_3 + \ldots$
  - ◊ You decide which game is most important
  - ◊ You want the last move (in normal play)
  - ◊ Analogy: Eating with a friend, want the last bite

Classification of Games

- Impartial
  - ◊ Same moves available to each player
  - ◊ Example: Nim
- Partisan
  - ◊ The two players have different options
  - ◊ Example: Domineering

Nim : The Impartial Game pt. I

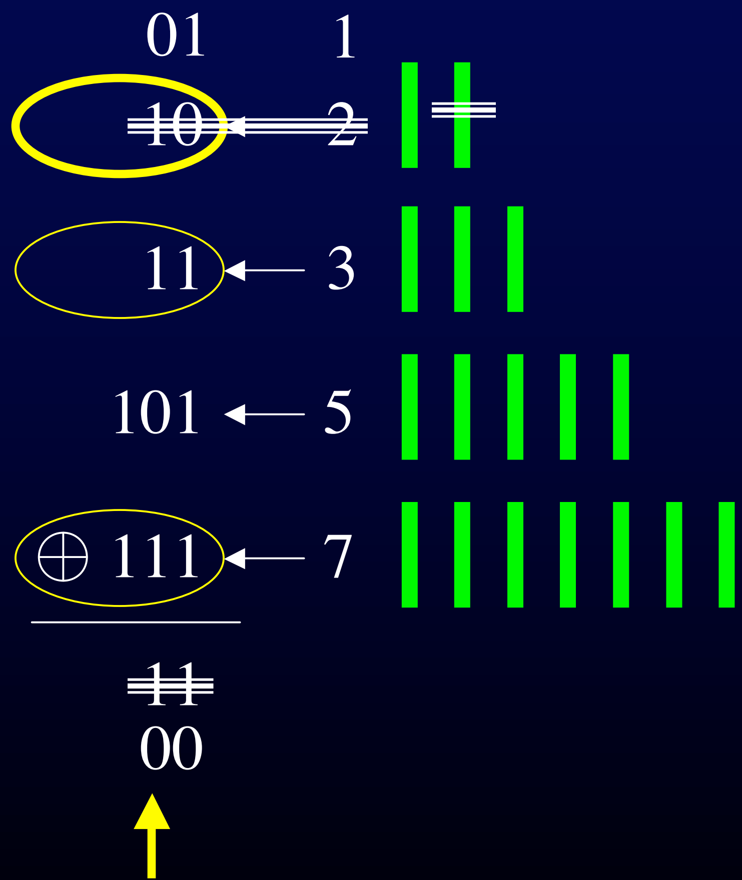
- Rules:
  - ◊ Several heaps of beans
  - ◊ On your turn, select a heap, and remove any positive number of beans from it, maybe all
- Goal
  - ◊ Take the last bean
- Example w/4 piles: (2,3,5,7)
- Who knows this game?

Nim: The Impartial Game pt. II

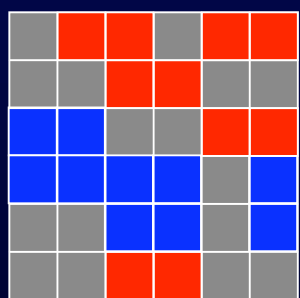
- Dan plays room in (2,3,5,7) Nim
- Ask yourselves:
  - ◊ Query:
    - First player win or lose?
    - Perfect strategy?
  - ◊ Feedback, theories?
- Every impartial game is equivalent to a (bogus) Nim heap

Nim: The Impartial Game pt. III

- Winning or losing?
  - ◊ Binary rep. of heaps
  - ◊ Nim Sum == XOR ⊕
  - ◊ Zero == Losing, 2nd P win
- Winning move?
  - ◊ Find MSB in Nim Sum
  - ◊ Find heap w/1 in that place
  - ◊ Invert all heap's bits from sum to make sum zero



Domineering: A partisan game

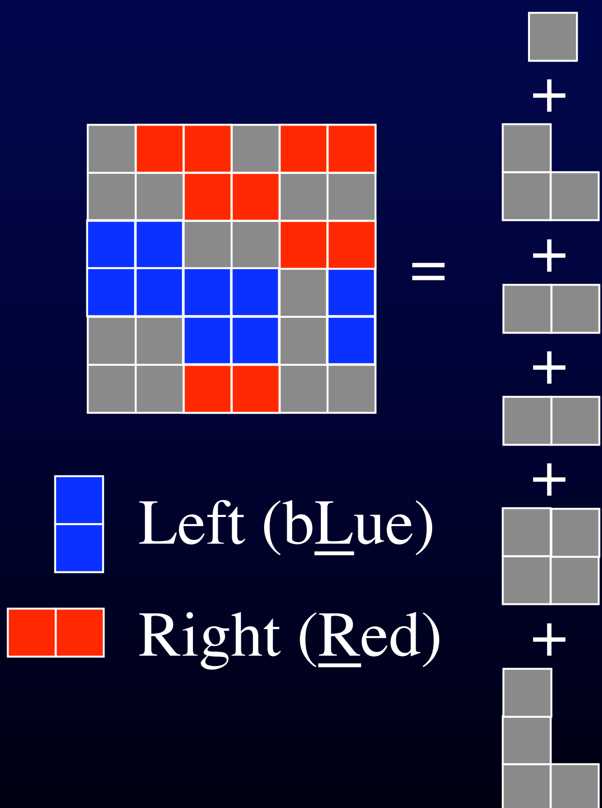


Left (bLue)

Right (Red)

- Rules (on your turn):
  - ◊ Place a domino on the board
  - ◊ Left places them North-South
  - ◊ Right places them East-West
- Goal
  - ◊ Place the last domino
- Example game
- Query: Who wins here?

Domineering: A partisan game



- Key concepts
  - ◊ By moving correctly, you guarantee yourself future moves.
  - ◊ For many positions, you want to move, since you can steal moves. This is a “hot” game.
  - ◊ This game decomposes into non-interacting parts, which we separately analyze and bring results together.

What do we want to know about a particular game?

- What is the value of the game?
  - ◊ Who is ahead and by how much?
  - ◊ How big is the next move?
  - ◊ Does it matter who goes first?
- What is a winning / drawing strategy?
  - ◊ To know a game's value and winning strategy is to have solved the game
  - ◊ Can we easily summarize strategy?

Combinatorial Game Theory The Basics I - Game definition

- A game, G, between two players, Left and Right, is defined as a pair of sets of games:
  - ◊ $G = \{G^L \mid G^R\}$
  - ◊ G^L is the typical Left option (i.e., a position Left can move to), similarly for Right.
  - ◊ G^L need not have a unique value
  - ◊ Thus if $G = \{a, b, c, \ldots \mid d, e, f, \ldots\}$, G^L means *a* or *b* or *c* or … and G^R means *d* or *e* or *f* or ...

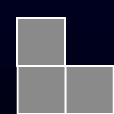
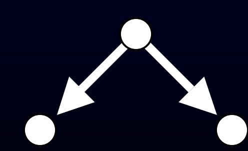
Combinatorial Game Theory
The Basics II - Examples: 0

- The simplest game, the Endgame, born day 0
  - ◊ Neither player has a move, the game is over
  - ◊ { Ø | Ø } = { | }, we denote by 0 (a number!)
  - ◊ Example of *P*, previous/second-player win, losing
  - ◊ Examples from games we've seen:

Nim	Domineering	Game Tree
		

Combinatorial Game Theory
The Basics II – Examples: *

- The next simplest game, * (“Star”), born day 1
  - ◊ First player to move wins
  - ◊ { 0 | 0 } = *, this game is not a number, it’s <u>fuzzy</u>!
  - ◊ Example of *N*, a next/first-player win, winning
  - ◊ Examples from games we’ve seen:

<u>Nim</u>	<u>Domineering</u>	<u>Game Tree</u>
1 		

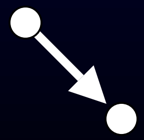
Combinatorial Game Theory
The Basics II - Examples: 1

- Another simple game, 1, born day 1
  - ◊ Left wins no matter who starts
  - ◊ { 0 | } = 1, this game is a number
  - ◊ Called a Left win. Partisan games only.
  - ◊ Examples from games we've seen:

Nim	Domineering	Game Tree
		

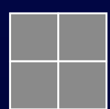
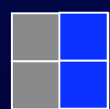
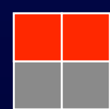
Combinatorial Game Theory
The Basics II - Examples: –1

- Similarly, a game, –1, born day 1
  - ◊ Right wins no matter who starts
  - ◊ $\{ \ | \ 0 \} = -1$, this game is a number.
  - ◊ Called a Right win. Partisan games only.
  - ◊ Examples from games we've seen:

Nim	Domineering	Game Tree
		

Combinatorial Game Theory
The Basics II – Examples

- Calculate value for Domineering game G:

G =  = {  |  }

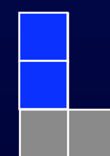
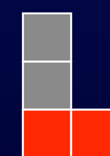
= { 1 | – 1 }

= ± 1

…this is a fuzzy hot value, confused with 0. 1st player wins.

 Left  Right

- Calculate value for Domineering game G:

G =  = {  ,  |  }

= { – 1 , 0 | 1 }

= { 0 | 1 }

= { .5 } (simplest #)

…this is a cold fractional value. Left wins regardless who starts.

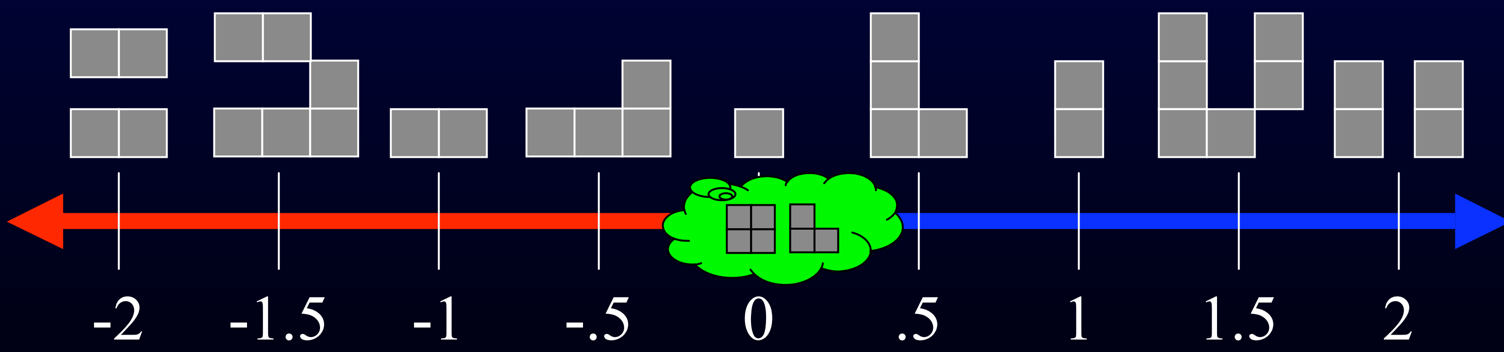
Combinatorial Game Theory
The Basics III – Outcome classes

- With normal play, every game belongs to one of four outcome classes (compared to 0):
  - ◊ Zero (=)
  - ◊ Negative (<)
  - ◊ Positive (>)
  - ◊ Fuzzy (||), incomparable, confused

		Right starts: and L has winning strategy	Right starts: and R has winning strategy
Left starts	and R has winning strategy	ZERO G = 0 2nd wins	NEGATIVE G < 0 R wins
Left starts	and L has winning strategy	POSITIVE G > 0 L wins	FUZZY G || 0 1st wins

Combinatorial Game Theory
The Basics IV – Values of games

- What is the value of a fuzzy game?
  - ◊ It's neither > 0, < 0 nor = 0, but confused with 0
  - ◊ Its place on the number scale is indeterminate
  - ◊ Often represented as a "cloud"



Combinatorial Game Theory The Basics V - Final thoughts

- There's much more!
  - ◊ More values
    - Up, Down, Tiny, etc.
  - ◊ How games add
  - ◊ Simplicity, Mex rule
  - ◊ Dominating options
  - ◊ Reversible moves
  - ◊ Number avoidance
  - ◊ Temperatures
- Normal form games
  - ◊ Last to move wins, no ties
  - ◊ Whose turn not in game
  - ◊ Rich mathematics
  - ◊ Key: Sums of games
  - ◊ Many (most?) games are not normal form!
    - What do we do then?
    - Computational GT!

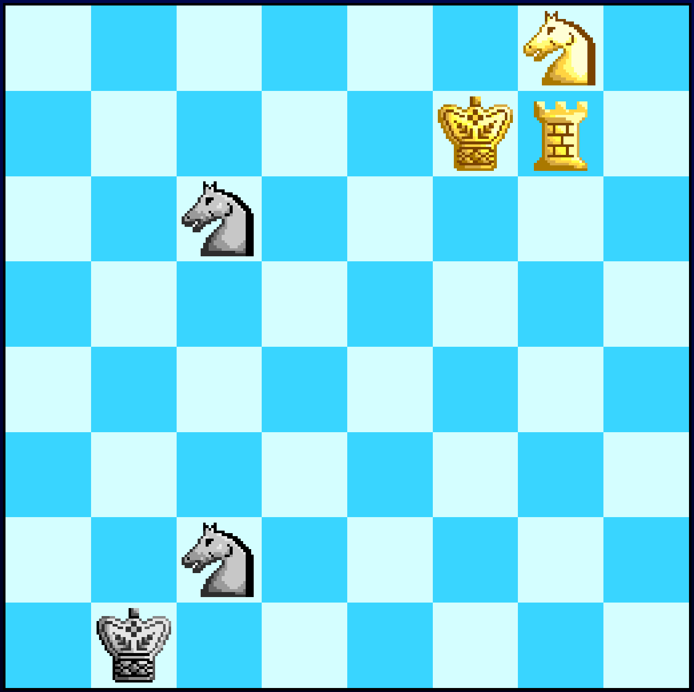
And now over to David for more Combinatorial examples…

Computational Game Theory (for non-normal play games)

- Large games
  - ◊ Can theorize strategies, build AI systems to play
  - ◊ Can study endgames, smaller version of original
    - Examples: Quick Chess, 9x9 Go, 6x6 Checkers, etc.
- Small-to-medium games
  - ◊ Can have computer solve and teach us strategy
  - ◊ I wrote a system called GAMESMAN which I use in CS0 (a SIGCSE 2002 Nifty Assignment)

How do you build an AI opponent for large games?

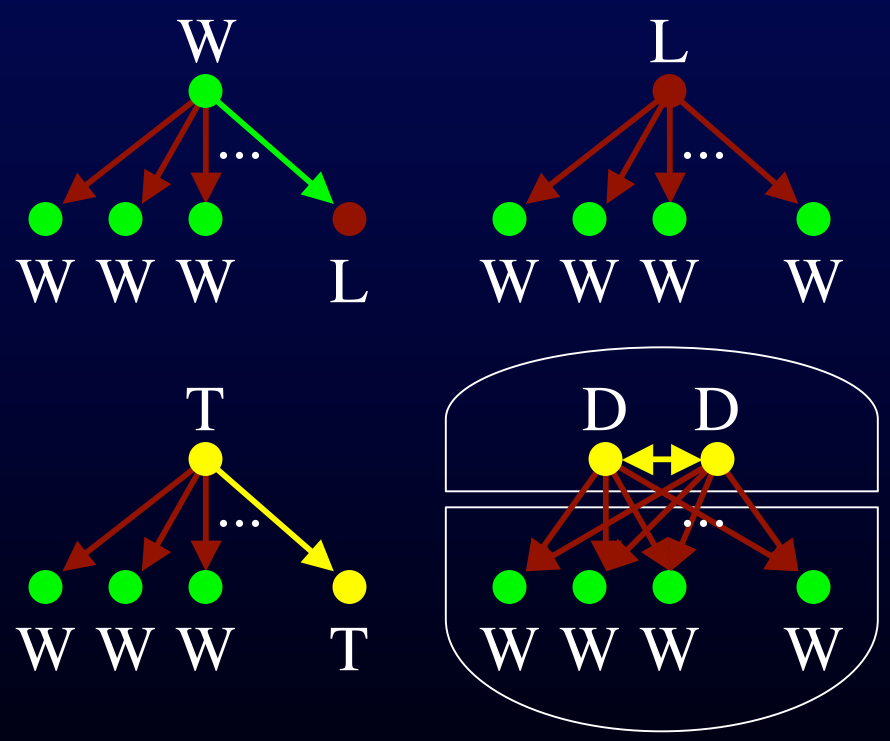
- For each position, create Static Evaluator
- It returns a number: How much is a position better for Left?
  - ◊ (+ = good, – = bad)
- Run MINIMAX (or alpha-beta, or A*, or …) to find best move



White to move, wins in move 243 with Rd7xNe7

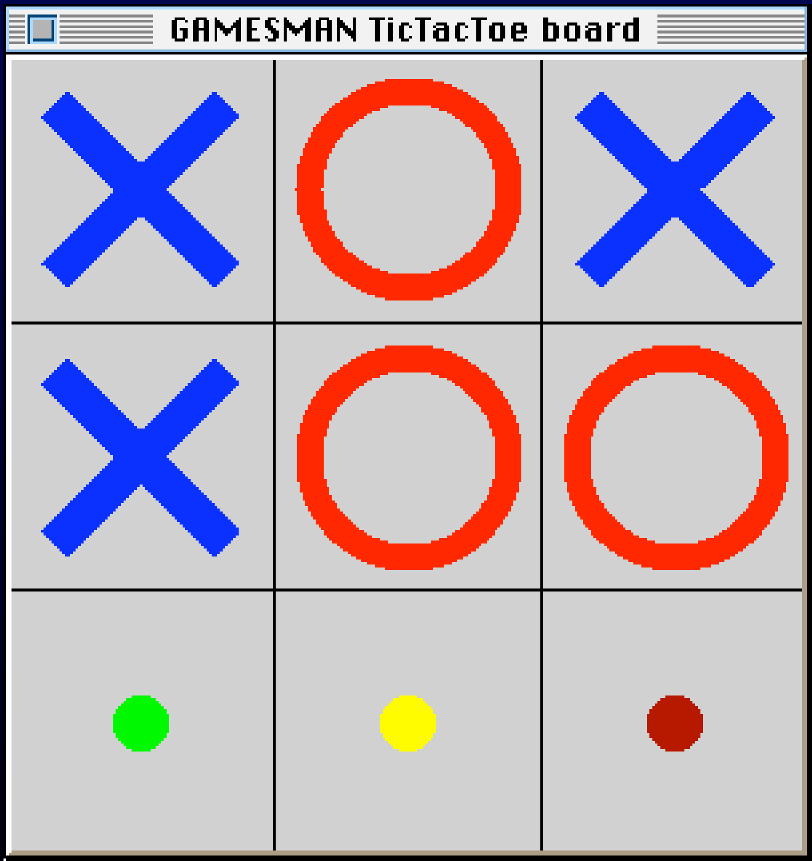
Computational Game Theory

- Simplify games / value
  - ◊ Store turn in position
  - ◊ Each position is (for player whose turn it is)
    - Winning (□ losing child)
    - Losing (All children winning)
    - Tieing (!□ losing child, but □ tieing child)
    - Drawing (can't force a win or be forced to lose)



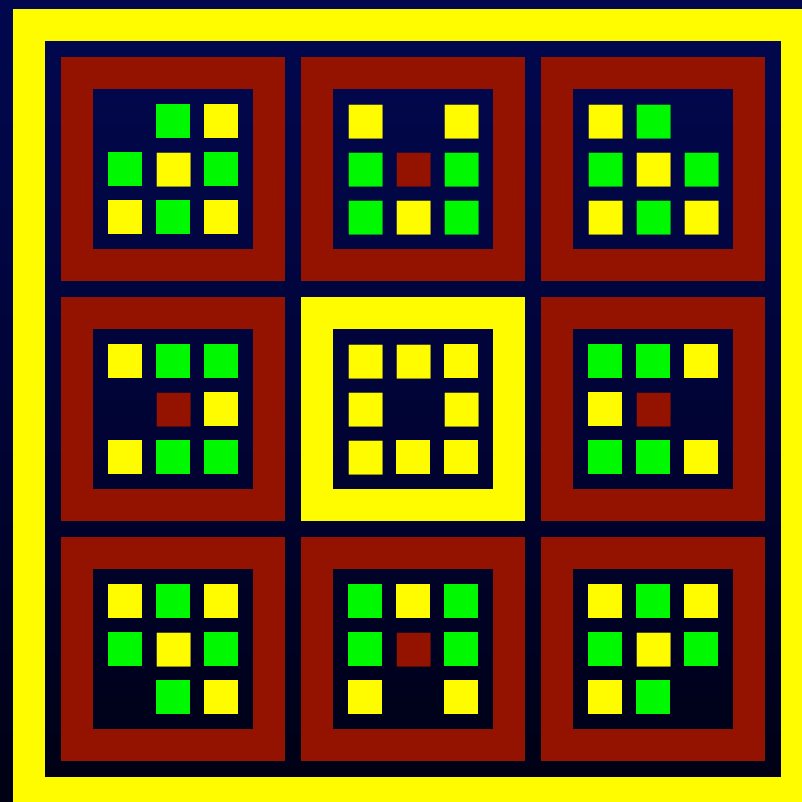
Computational Game Theory
Tic-Tac-Toe

- Rules (on your turn):
  - ◊ Place your X or O in an empty slot
- Goal
  - ◊ Get 3-in-a-row first in any row/column/diag.
- Misére is tricky



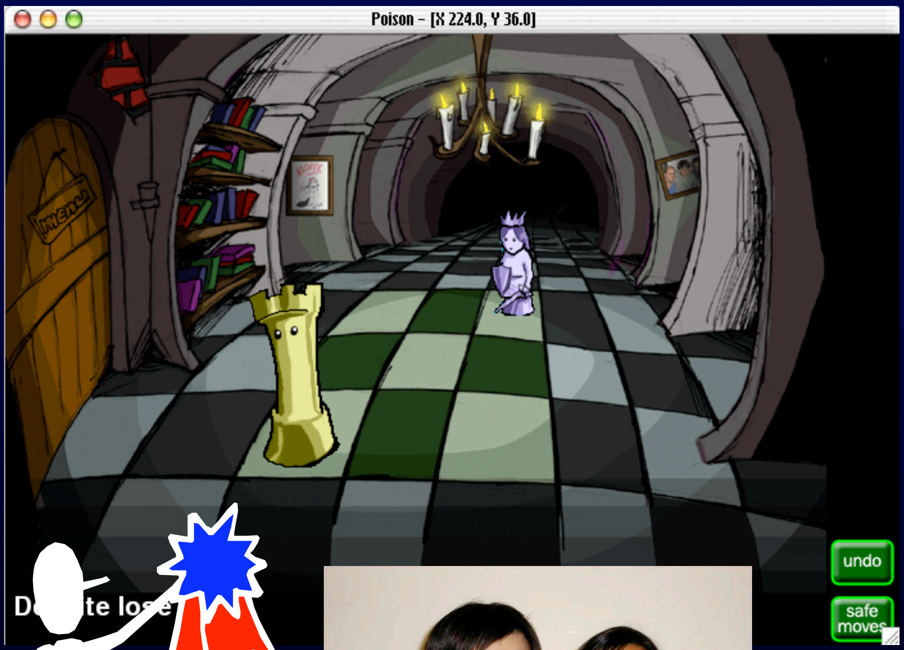
Computational Game Theory
Tic-Tac-Toe Visualization

- Visualization of values
- Example with Misére
  - ◊ Outer rim is position
  - ◊ Next levels are values of moves to that position
  - ◊ Recursive image
  - ◊ Legend: Lose, Tie, Win



Use of games in projects (CS0) Language: Scheme & C

- Every semester…
  - ◊ New games chosen
  - ◊ Students choose their own graphics & rules (I.e., open-ended)
  - ◊ Final Presentation, best project chosen, prizes
- Demonstrated at SIGCSE 2002 Nifty Assignments



And now over to Peter…

- Two player games
- More motivation
- Prisoner's Dilemma

Summary

- Games are wonderful pedagogic tools
  - ◊ Rich, colorful, easy to state problems
  - ◊ Useful in lecture *or* for homework / projects
  - ◊ Can demonstrate so many CS concepts
- We've tried to give broad theoretical foundations & provided some nuggets…

Resources

- www.cs.berkeley.edu/~ddgarcia/eyawtkagtbwata/
- www.cut-the-knot.org
- E. Berlekamp, J. Conway & R. Guy: *Winning Ways I & II* [1982]
- R. Bell and M. Cornelius: *Board Games around the World* [1988]
- K. Binmore: *A Text on Game Theory* [1992]