The Complexity of Accurate Floating Point Computation

or

Can we do Numerical Linear Algebra In Polynomial Time?

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Goals

- Def: Accurate floating point (FP) computation means with guaranteed relative error < 1
  - $10^{-2} \equiv 2$ digits, $10^{-16} \equiv 16$ digits, ...
  - zero must be exact
- Def: Efficient computation of an expression means in time poly(size of the expression, size of the input)
- Def: CAE means "compute accurately and efficiently"
- Goal: Understand cost of accurate FP computation
  - What FP expressions can we CAE?
  - Are there FP expressions that we cannot CAE?
  - For what structured matrices
    (i.e. with FP expressions as entries)
    are there matrix computations that we can CAE?
    - LU, QR, Inv, Pinv, SVD, Eig, ...

Structure of Results (1)

- Classes of FP expressions/matrices that we can CAE depends strongly on Model of FP Arithmetic

  1. Traditional ("$1 + \delta$") Model (TM for short):
     $fl(a \otimes b) = (a \otimes b)(1 + \delta)$, $|\delta| \le \epsilon \ll 1$
     no over/underflow
  2. Bit model: inputs are $f \cdot 2^e$, with "large exponents" (LEM for short):
     "natural" model for algorithms, analysis
  3. Bit model: inputs are $f \cdot 2^e$, with "small exponents" (SEM for short):
     integers in disguise, well understood
  4. Others have been proposed (not today)
     (a) Blum/Shub/Smale
     (b) Cucker/Smale
     (c) Pour-El/Richards

Structure of Results (2)

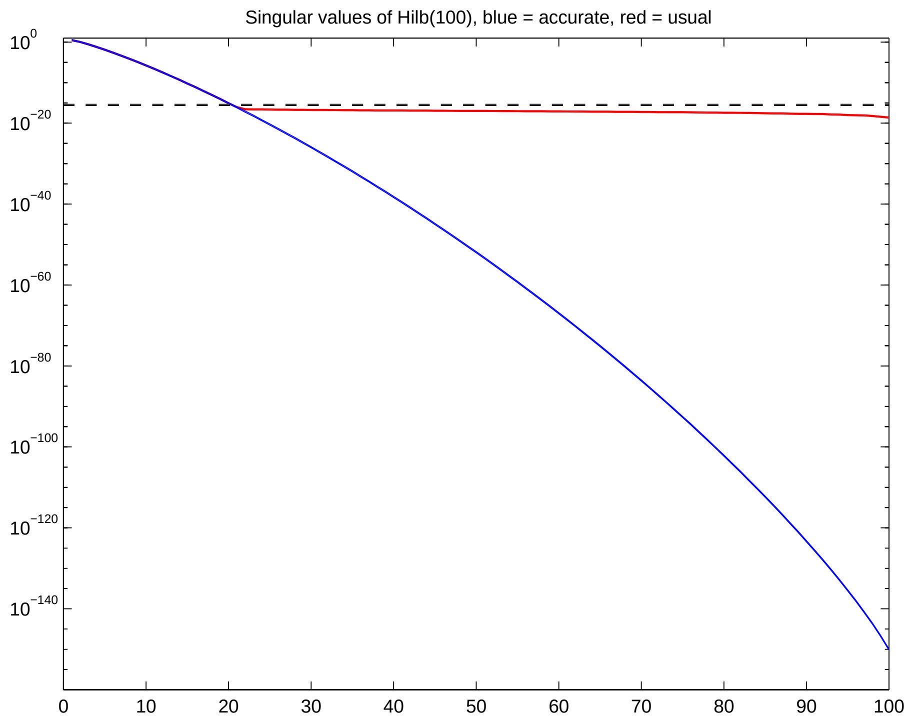
- Classes of expressions (matrices) that we can CAE are described by factorizability properties of expressions (minors of matrices)

$$\mathbf{TM} \underset{\neq}{\subseteq} \mathbf{LEM} \underset{\neq ?}{\subseteq} \mathbf{SEM}$$

- New algorithms can be exponentially faster than conventional algorithms that just use high enough precision
- Cost(CAE in LEM) $\geq$ Cost("symbolic computing")
- Many recent results (see Koev's talk too)

Example: 100 by 100 Hilbert Matrix $H(i,j) = 1/(i+j-1)$

- Eigenvalues range from 1 down to 10^{-150}
- Old algorithm, New Algorithm, both in 16 digits



- $D = \log(\lambda_1/\lambda_n) = \log \mathrm{cond}(A)$ (here $D = 150$ digits)
- Cost of Old algorithm in high enough precision $= O(n^3 D^2)$
- Cost of New algorithm $= O(n^3)$ - independent of cond(A)

Central Role of Minors

- Being able to CAE det(A) is necessary for CAE
  - $A = LU$ with pivoting
  - $A = QR$
  - Eigenvalues λ_i of A
  - Related factorizations ...
    - * Proof: $\det(A) = \pm \prod_i U_{ii} = \pm \prod_i R_{ii} = \prod_i \lambda_i = \cdots$
- Being able to CAE all minors of A is sufficient for CAE
  - A^{-1}
    - * Proof: Cramer's rule
    - * Only need $n^2 + 1$ minors
  - $A = LU$ or $A = LDU$ with pivoting
    - * Proof: Each entry of L, D, U a quotient of minors
    - * Only need $O(n^2)$ or $O(n^3)$ minors
  - Singular values
    - * Proof: Rank-revealing $A = LDU$, then SVD of LDU
- Similar result for QR, pseudoinverse via $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$, etc.
- Examine which expressions (minors) we can CAE

Accurate SVD of any rank-revealing $A = XDY^T$

- SVD is $A = U\Sigma V^T$
- Many accurate algorithms, here is simplest:
  1. Compute SVD of $DY^T = U_1 \Sigma_1 V_1^T$ using one-sided Jacobi
  2. Multiply $W = XU_1$
  3. Compute SVD of $W\Sigma_1 = U\Sigma V_2^T$ using one-sided Jacobi
  4. Multiply $V = V_1 V_2$
- To guarantee efficiency, find eigenvalues of

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} = \frac{1}{2^{1/2}} \cdot \begin{bmatrix} L & L \\ U^T & -U^T \end{bmatrix} \cdot \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} \cdot \begin{bmatrix} L^T & U \\ L^T & -U \end{bmatrix} \cdot \frac{1}{2^{1/2}}$$
$$\equiv Z \cdot \hat{D} \cdot Z^T$$

  by performing bisection on $\lambda \hat{D} - Z^{-1} Z^{-T}$
- Relative error $= O(\kappa(X) \cdot \kappa(Y))$

Why roundoff is harmless

- We want $A = U\Sigma V^T$ where $\Sigma = \mathrm{diag}(\sigma_1, ..., \sigma_n)$
- But we compute $\bar{A} = \bar{U}\bar{\Sigma}\bar{V}^T$ where $\bar{\Sigma} = \mathrm{diag}(\bar{\sigma}_1, ..., \bar{\sigma}_n)$

Absolute (additive) Perturbations vs. Relative (multiplicative) Perturbations

$$\bar{A} = A + \sigma_{\max} \cdot E \qquad\qquad \bar{A} = (I + E)A$$

$$\|E\| \ll 1$$

$$|\sigma_i - \bar{\sigma}_i| \le \|E\| \cdot \sigma_{\max} \qquad\qquad |\sigma_i - \bar{\sigma}_i| \le \|E\| \cdot \sigma_i$$

How do we CAE $A = L \cdot D \cdot U$ for a Hilbert (or Cauchy) Matrix?

- How can we lose accuracy in computing?
  - TM: $fl(a \otimes b) = (a \otimes b)(1 + \delta)$, $|\delta| \leq \epsilon \ll 1$
  - OK to multiply, divide, add positive numbers
  - OK to subtract exact numbers (initial data)
  - Cancellation when subtracting approximate results dangerous:

$$\begin{array}{rl} & .12345\text{xxx} \\ - & .12345\text{yyy} \\ \hline & .00000\text{zzz} \end{array}$$

- Cauchy: $C(i,j) = 1/(x_i + y_j)$
- Fact 1: $\det(C) = \Pi_{i<j}(x_j - x_i)(y_j - y_i)/\Pi_{i,j}(x_i + y_j)$ - No bad cancellation
- Fact 2 : Each minor of C also Cauchy
- Fact 3 : Each entry of L, D, U is a (quotient of) minors
- Change inner loop of Gaussian Elimination from

$$C(i,j) := C(i,j) - C(i,k) * C(k,j)/C(k,k)$$

  to

$$C(i,j) := C(i,j) * (x_i - x_k) * (y_j - y_k)/(x_k + y_j)/(x_i + y_k)$$

- Each entry of L, D, U accurate to most digits!

Cost of Accuracy (1)

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy										
TP Cauchy										
Vandermonde										
TP Vandermonde										
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

NENP = Neville Elimination (bidiagonal factorization) with No Pivoting

$Ax = b$ Forw. = solving with small forward error: $|x - \hat{x}| \leq O(\epsilon)|A^{-1}| \cdot |b|$

$Ax = b$ Backw. = solving with small backward error: $\max_i \frac{|A\hat{x}-b|_i}{(|A||\hat{x}|+|b|)_i} = O(\epsilon)$

Cost of Accuracy (2)

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	$C_{ij} = 1/(x_i + y_j)$									
TP Cauchy	$x_i \nearrow$, $y_j \nearrow$, $x_1 + y_1 > 0$									
Vandermonde	$V_{ij} = x_i^{j-1}$, x_i distinct									
TP Vandermonde	$0 < x_i \nearrow$									
Confluent Vandermonde	if some x_i coincide, differentiate rows of V									
TP Confluent Vandermonde	$0 < x_i \nearrow$									
Vandermonde 3 Term Orth. Poly.	$V_{ij} = P_j(x_i)$, P_j a j-th orthogonal poly. in 3-term recurrence									
Same + other cond.	$0 < x_i \nearrow$, positivity conditions on 3-term recurrence									
Generalized Vandermonde	$G_{ij} = x_i^{\lambda_j + j - 1}$, λ_j nonnegative increasing integer sequence									
TP Generalized Vandermonde	$0 < x_i \nearrow$									

TP = Totally Positive (all minors nonnegative)

Cost of Accuracy (3)
Known results of others

Type of Matrix	det(A)	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3		n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3		n^2	n^2	n^2
Vandermonde										
TP Vandermonde										
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Exploit $\det(C) = \Pi_{i<j}(x_j - x_i)(y_j - y_i) / \Pi_{ij}(x_i + y_j)$

Cost of Accuracy (4)
Known results of others + New Results

Type of Matrix	det(A)	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde										
TP Vandermonde										
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Do GECP, apply new SVD algorithm

Cost of Accuracy (5)
Known results of others

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax=b$ Forw.	$Ax=b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2							n^2		
TP Vandermonde	n^2	n^3		n^2	n^2			n^2	n^2	n^2
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Cost of Accuracy (6)
Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2						n^3	n^2		
TP Vandermonde	n^2	n^3		n^3			n^3	n^2	n^2	n^2
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Vandermonde = Cauchy $\times$ DFT

Cost of Accuracy (7)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax=b$ Forw.	$Ax=b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2						n^3	n^2		
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Use new alg for Generalized Vandermonde ...

Cost of Accuracy (8)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax=b$ Forw.	$Ax=b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde										
TP Confluent Vandermonde										
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Can't add $x+y+z$ in TM

Cost of Accuracy (9)
Known results of others

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2							n^2		
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Cost of Accuracy (10)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	No	
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.										
Same + other cond.										
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: Can't add $x + y + z$ in TM

Cost of Accuracy (11)

Known results of others

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	No	
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.	n^2									
Same + other cond.	n^2	n^3							n^2	
Generalized Vandermonde										
TP Generalized Vandermonde										

Cost of Accuracy (12)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax = b$ Forw.	$Ax = b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	No	
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3			
Same + other cond.	n^2	n^3					n^3		n^2	
Generalized Vandermonde										
TP Generalized Vandermonde										

Proof: See Koev's talk

Cost of Accuracy (13)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax=b$ Forw.	$Ax=b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	No	
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3			
Same + other cond.	n^2	n^3					n^3		n^2	
Generalized Vandermonde	No	No	No	No	No	No	No	No	No	
TP Generalized Vandermonde										

Proof: Can't add $x + y + z$ in TM

Cost of Accuracy (14)

Known results of others + New Results

Type of Matrix	$\det(A)$	A^{-1}	Any minor	GE NP	GE PP	GE CP	SVD	NE NP	$Ax=b$ Forw.	$Ax=b$ Backw.
Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	
TP Cauchy	n^2	n^2	n^2	n^3	n^3	n^3	n^3	n^2	n^2	n^2
Vandermonde	n^2	No	No	No	No	No	n^3	n^2	No	
TP Vandermonde	n^2	n^3	exp	n^2	n^2	exp	n^3	n^2	n^2	n^2
Confluent Vandermonde	n^2	No	No	No	No	No		n^2	No	
TP Confluent Vandermonde	n^2	n^3		n^3				n^2	n^2	n^2
Vandermonde 3 Term Orth. Poly.	n^2						n^3			
Same + other cond.	n^2	n^3					n^3		n^2	
Generalized Vandermonde	No	No	No	No	No	No	No	No	No	
TP Generalized Vandermonde	$\Lambda n + n^2$	$\Lambda n^2 + n^3$	exp	Λn^2	Λn^2	exp	exp	Λn^2	Λn^2	Λn^2

$G_{ij} = x_i^{\lambda_j + j - 1}$, $0 \le \lambda_i \nearrow$

$\Lambda = (\lambda_1 + 1) \cdot (\lambda_2 + 1)^2 \cdots (\lambda_n + 1)^2$

(For Proof, see Koev's PhD thesis)

Previous best algorithm: $n^{\lambda_1 + \cdots + \lambda_n}$

Other examples in Traditional $1 + \delta$ model

- Diagonal * Totally Unimodular (TU) * Diagonal
  - TU $\Leftrightarrow$ each minor $\in \{0, \pm 1\}$
  - Poincaré: Signed incidence matrix on graph $\Rightarrow$ TU
  - Includes 2nd centered difference approximations to Sturm-Liouville equations and elliptic PDEs on uniform meshes
  - One-line change to GE makes it accurate
- M-Matrices
  - Store as off-diagonals, nonnegative row sums
  - See Koev's PhD thesis
- Sparse matrices with
  - Acyclic sparsity patterns, Cost $= O(n^3)$
  - Particular sparsity and sign patterns ("Total Sign Compound") Cost $= O(n^4)$
- Other Totally Positive matrices (but cost not always poly)

- What do these matrices have in common?

Traditional $1+\delta$ Model - What we can do

- Goal: evaluate homogeneous polynomial $f(x)$ accurately on $\mathcal{D}$
- Property A: $f = \Pi_m f_m$ where each factor f_m satisfies one of
  - f_m of the form x_i, $x_i - x_j$ or $x_i + x_j$
  - $|f_m|$ bounded away from 0 on $\mathcal{D}$
- Conjecture 1: f satisfies Prop. A iff $f(x)$ can be evaluated accurately
- Conjecture 2: f satisfies Prop. A iff $f(x)$ has a *relative perturbation theory*:
  - relative error in output = O($\kappa_{rel}\cdot$ relative error in input)
  - $\kappa_{rel} = 1/\min \frac{|x_i \pm x_j|}{|x_i|+|x_j|}$ = 1 / smallest relative gap among inputs
  - Tiny outputs often well conditioned
    * Smallest eigenvalues often desired
    * Relative perturbation theory justifies computing them!
- Intuition: Everything works if $f(x)$ has factors only of forms
  - x_i
  - $x_i \pm x_j$
  - positive stuff

Otherwise, $\forall$ algorithms $\exists$ roundoff errors that make relative error large

Bit Models of FP Arithmetic

- Inputs of form $f \cdot 2^e$, e and f integers
- size(X) = # bits used to represent X
- size($f \cdot 2^e$) = #bits(f) + #bits(e)
- Can evaluate any rational expression accurately
  - Convert to poly/poly, using high enough precision
  - Question is cost
- Cost depends strongly on # exponent bits
  - Small Exponent Model (SEM)
    - #bits(e) = O(log(#bits(f)))
    - Equivalent to integer arithmetic
    - Can CAE many problems
  - Large Exponent Model (LEM)
    - #bits(e) and #bits(f) independent
    - "Natural" model for algorithm design
    - Algorithms work for any input magnitudes

Differences between SEM and LEM - 1

- Recall definitions for size of $f \cdot 2^e$
  - Small Exponent Model (SEM): $\#\text{bits}(e) = O(\log(\#\text{bits}(f)))$
  - Large Exponent Model (LEM): $\#\text{bits}(e)$, $\#\text{bits}(f)$ independent
- SEM and "integer arithmetic" equivalent
  - Represent $f \cdot 2^e$ as integer, not pair (f, e)
  - $\#\text{bits}(f \cdot 2^e) = \#\text{bits}(f) + e \approx \#\text{bits}(f) + 2^{\#\text{bits}(e)} = \text{poly}(\#\text{bits}(f))$
- LEM and "integer arithmetic" not equivalent
  - $2^{\#\text{bits}(e)}$ exponentially larger than $\#\text{bits}(e)$
- # bits in FP expressions much bigger for LEM than SEM
  - SEM: $\text{size}(x \cdot y) \le \text{size}(x) + \text{size}(y)$
  - LEM: $\text{size}(x \cdot y) \le \text{size}(x) \cdot \text{size}(y)$
  - The product of two n-bit numbers:

$$\begin{aligned}
\text{size}(x \cdot y) &= \text{size}\Big(\sum_{i=1}^{n} 2^{r_i} \cdot \sum_{j=1}^{n} 2^{s_j}\Big) \\
&= \text{size}\Big(\sum_{i,j=1}^{n} 2^{r_i + s_j}\Big) \\
&= n^2 \text{ bits}, \text{not } 2n \text{ bits}
\end{aligned}$$

Differences between SEM and LEM - 2

- Recall definitions for size of $f \cdot 2^e$
  - Small Exponent Model (SEM): #bits(e) = O(log(#bits(f)))
  - Large Exponent Model (LEM): #bits(e), #bits(f) indep.
- Cond(A) in LEM can be exponentially larger than in SEM
  - SEM: $\log \mathrm{cond}(A)$ is poly(size(A))
    - Conventional algorithms using $\log \mathrm{cond}(A)$ bits are polynomial
  - LEM: $\log \mathrm{cond}(A)$ can be exponential in size(A)
    - $\kappa(\mathrm{diag}(2^e, 1)) = 2^e \approx 2^{2^{\#\mathrm{bits}(e)}}$
    - Conventional algorithms using $\log \mathrm{cond}(A)$ bits are not polynomial
- $\log \log \mathrm{cond}(A)$ is lower bound on complexity of any FP algorithm
  - # bits needed to print out exponent of answer

Differences between SEM and LEM - 3

- Recall definitions for size of $f \cdot 2^e$
  - Small Exponent Model (SEM): #bits(e) = O(log(#bits(f)))
  - Large Exponent Model (LEM): #bits(e), #bits(f) indep.
- Determinant of any SEM matrix computable exactly in poly time
  - Put all $A_{ij}(x) = P_{ij}(x)/Q_{ij}(x)$ over common denominator
  - Compute each numerator, denominator exactly
  - Compute determinant accurately in poly time using Clarkson's Alg.
  - Can do accurate linear algebra in poly time
- Getting arbitrary bit of expression in LEM very hard
  - Getting arbitrary bit of $\Pi_{i=1}^{n}(1 + x_i)$ is as hard as permanent(A)
  - A null vector of singular matrix of LEM floats can have exponentially many bits
    - Testing singularity may not even be in NP!
  - Matrices need structure to CAE

How to CAE $S = \sum_{i=1}^{n} s_i$

- Suppose
  - All s_i have m-bit fractions, normalized
  - One M-bit register S available ($M > m$)
  - Let $\bar{n} = 1 + 2^{M-m} + 2^{M-2m} + \cdots + 2^{M \bmod m}$
- Algorithm for $\sum_{i=1}^{n} s_i$:
  1. Sort so $\mathrm{EXP}(s_1) \geq \mathrm{EXP}(s_2) \geq \cdots \geq \mathrm{EXP}(s_n)$
  2. $S = 0$; for $i = 1$ to n do $S = S + s_i$ ... round to nearest
- Theorem (D., Hida): Exactly one error bound below applies, is attainable:
  1. If $n \leq \bar{n}$ then relative error in S at most 1 ulp
  2. If $n = \bar{n} + 1$ and $M \geq 2m$ then relative error in S at most 2 ulps
  3. If $n = \bar{n} + 1$ and $M < 2m$ then relative error in S at most 2^{2m-M} ulps
  4. If $n \geq \bar{n} + 2$ then relative error in S can be arbitrary
- Example: $S = \sum_{i=1}^{n} x_i \cdot y_i$, $m = 2 \cdot 24 = 48$, $M = 53 \rightarrow \bar{n} = 2^5 + 1 = 33$
- Example: $S = \sum_{i=1}^{n} x_i \cdot y_i$, $m = 2 \cdot 24 = 48$, $M = 64 \rightarrow \bar{n} = 2^{16} + 1 = 65537$
- Sorting can be mostly eliminated
- Priest, Knuth, Kahan, Bohlender, Dekker, Pichat, ...

Which FP Expressions can we CAE in the Large Exponent Model (LEM)?

- Def: $r(x)$ is in factored form if

$$r(x) = \prod_{i=1}^{n} p_i(x_1, ..., x_k)^{e_i}$$

where

$$p_i(x_1, ..., x_k) = \sum_{j=1}^{t} \alpha_{ij} \cdot x_1^{e_{ij1}} \cdots x_k^{e_{ijk}}$$

and

$$\text{size}(r) = \#\text{bits to write down } r$$

- Theorem: We can CAE r in time poly(size(r))
  - Compute each monomial $\alpha_{ij} \cdot x_1^{e_{ij1}} \cdots x_k^{e_{ijk}}$ exactly
  - Compute $p_i(x_1, ..., x_k)$ by sorting and adding monomials
  - Compute $p_i(x_1, ..., x_k)^{e_i}$ by repeated squaring, rounding
  - Compute $\prod_{i=1}^{n} p_i(x_1, ..., x_k)^{e_i}$ by multiplying, rounding
- Def: A family $A_n(x)$ of n-by-n rational matrices is polyfactorable if each minor $r(x)$ is in factored form of size size(r) $= O(\text{poly}(n))$
- Thm: Suppose $A_n(x)$ is polyfactorable. Then in the LEM we can CAE LU with pivoting, A^{-1}, singular values.

Differences between Models

- **What can we CAE in LEM that we could not in TM?**
  - **Rational Expressions**
    - **LEM: anything polynomial in size (in factored form) can be computed accurately in polynomial time**
    - **(*Not* det A where each A_{ij} independent: size is $n!$)**
    - **TM: Constraints on zero/pole set: inside** $\mathcal{A} = \cup_i \{x_i = 0\} \;\cup\; \cup_{i \neq j} \{x_i = x_j\} \;\cup\; \cup_{i \neq j} \{x_i = -x_j\}$
  - **Matrix computations**
    - **LEM: Take any $A(x)$ that we can CAE in TM, substitute $x_i = p_i(y)$**
    - **Green's matrices (inverses of tridiagonals, represented as $A_{ij} = x_i y_j$)**
- **What can we CAE in SEM that we could not in LEM?**
  - **Rational matrices with arbitrary poly-sized entries**

Cost comparison of LEM to symbolic algebra

- Cost(Accurate evaluation in Large Exponent Model) $\geq$ Cost(symbolic expression $\equiv 0$?)
- Proof idea: Simulate symbolic algebra using large exponents
  - 2^a, 2^b like x and y, because we can extract a and b from $2^a \cdot 2^b = 2^{a+b}$
  - Cost(Accurate evaluation of p) $\geq$ Cost($p \equiv 0$?)
  - Suppose $p(x_0, ..., x_{m-1})$ a symbolic polynomial with max degree $D-1$, integer coeffs of max # bits $B-1$
  - Let $X_i = 2^{B \cdot D^i}$
  - Then $p(X_0, ..., X_{m-1}) = 0$ iff $p \equiv 0$
    * Idea: bits in typical term $\alpha \cdot x_1^{e_1} \cdots x_{m-1}^{e_{m-1}}$ of p do not "overlap" so cannot cancel in sum
- Example: determinant of A each entry of which is rational

Open Questions

- Are there FP expressions that we provably cannot CAE in LEM?
  - $\Pi_{i=1}^{n}(1 + x_i) - \Pi_{j=1}^{n}(1 + y_j)$
  - Determinant of general matrix
  - Determinant of tridiagonal matrix
  - Such an example could distinguish LEM from SEM
- What does symbolic computing complexity tell us about complexity in LEM?
- What changes if we have sign information?
  - We have accurate algorithms for all TP matrices, but not efficient
  - How big a class of TP matrices can we do efficiently?
- Differential equations
  - Only simplest ones understood (M-matrices)
  - What about other discretizations?
  - Conjecture: Accuracy depends only on geometry, not material properties
- Relationship to graph-based preconditioners of Vaidya et al
- Exploit sparsity
- What about nonsymmetric eigenproblem?

Conclusions

- **We have identified many classes of floating point expressions and matrix computations that permit**
  - **Accurate solutions: relative error < 1**
  - **Efficient solutions: time = poly(input size)**
- **Explored 3 natural models of arithmetic: TM, LEM, SEM**
  - **New efficient algorithms for each**
  - **TM $\subsetneq$ LEM $\subsetneq ?$ SEM**
- **Lots of open problems**
- **Reports available**
  - **Upcoming ICM paper at** `www.cs.berkeley.edu/~demmel/ICM_final.ps`
  - **Koev's UC Berkeley PhD thesis at** `www.math.berkeley.edu/~plamen/a.ps`
  - **Accurate FP Addition,** `www.cs.berkeley.edu/~demmel/AccurateSummation.ps`
  - **D + Koev paper on LEM in Structured Matrices in Math, CS and Eng II, AMS, 2001 (**`www.cs.berkeley.edu/~demmel/NASC.ps`**)**
  - **SIMAX, v. 21, n. 2, pp 562–580, 1999**
  - **Lin. Alg. Appl., vol 299, issue 1-3, pp 21–80, 1999**
  - **These slides:** `www.cs.berkeley.edu/~demmel/HH02.ps`

Postdocs Available!

- **One to work on Parallel Eigensolvers (Holy Grail)**
- **One to work on Parallel Direct Solvers (SuperLU)**
- **See me or email: demmel@cs.berkeley.edu**

There was a numerical analyst from Nantucket ...

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Who sorted his exponents with a bucket.

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When asked ”How do you cope
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There was a numerical analyst from Nantucket
Who sorted his exponents with a bucket.
When asked ”How do you cope
With huge exponent scope?”
Said: “When I see such a problem, I duck it!”