CS 70 Discrete Mathematics and Probability Theory
Spring 2011 Demmel HW 1

Due Jan 27, 11:59pm

You may work in groups of up to 3 people (no larger!). Please read the group collaboration policies on bSpace or `www.cs.berkeley.edu/~demmel/cs70_Spr11` before beginning group work. You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please begin your answer to each question on a new sheet of paper, and make sure that each sheet is labeled with your name, section number, GSI name, the assignment number, the question number, and "CS70–Spring 2011".

Turn in each question in a different box in 283 Soda Hall: Question 1 in the box labeled "CS70 - 1", Question 2 in the box labeled "CS70 - 2", etc. Reason: Different problems will be graded in parallel by different readers.

Warning: You risk receiving no credit, or losing points, for any homework that does not conform to the above regulations! Please take the time to write clear and concise solutions; we will not grade messy or unreadable submissions. No late homeworks will be accepted. We will drop the lowest two homework scores.

1. (10 pts.) Following Directions; Collaboration policy.
Part 1. You get an automatic 5 points for following directions (e.g., turning your question into the right box, and labeling every sheet of paper with all requested information).

Part 2. Two CS 70 students Alice and Bob decide to work in a group. They collaborate to figure out how to solve every question on the homework. Then, they split up the questions: Alice writes down the answers they came up with for questions 1–4, Bob writes down the answers they came up with for questions 5–7, and then they swap these papers and use them to finish their writeups. Under the CS 70 collaboration policy, is this OK, assuming they worked together on figuring out how to solve each question?

2. (12 pts.) If you show up on time, you won't have to work this hard!
You show up late to CS 70 lecture and come in the middle of a complex derivation involving the propositions P, Q, and R. From what you can see on the board, you're able to deduce that the following three propositions are true: $P \implies \neg P$, $Q \implies R$, $P \vee Q \vee \neg R$. Unfortunately, it looks like the definitions of the propositions P, Q, and R have already been erased.

1. Do you have enough information to deduce the truth value of P? If yes, what is the truth value of P?
2. Do you have enough information to deduce the truth value of Q? If yes, what is the truth value of Q?
3. Jim asks the class whether $(\neg Q \wedge R) \vee (Q \wedge \neg R)$ is true. Do you have enough information to deduce the truth value of this proposition? If yes, what is its truth value?

3. (20 pts.) Practice with quantifiers.
Part 1. Which of the following propositions is true? ($\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of natural numbers, and $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ denotes the set of integers.)

1. $(\forall x \in \mathbb{N})(x^2 < 9) \implies (\forall x \in \mathbb{N})(x^2 < 10)$.
2. $(\forall x \in \mathbb{N})(x^2 < 10) \implies (\forall x \in \mathbb{N})(x^2 < 9)$.
3. $(\forall x \in \mathbb{N})(x^2 < 9 \implies x^2 < 10)$.
4. $(\forall x \in \mathbb{N})(x^2 < 10 \implies x^2 < 9)$.
5. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x^2 < y)$.
6. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(x^2 < y)$.
7. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x^2 < y \implies x < y)$.
8. $(\exists y \in \mathbb{N})(\forall x \in \mathbb{N})(x^2 < y \implies x < y)$.
9. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x < y \implies x^2 < y)$.
10. $(\exists y \in \mathbb{N})(\forall x \in \mathbb{N})(x < y \implies x^2 < y)$.

Part 2. Rewrite the following quantified proposition in an equivalent form with all negations ("$\neg$", "$\neq$") removed.

$$\neg(\forall i \in \mathbb{N})\neg(\exists j \in \mathbb{N})(\exists k \in \mathbb{N})\neg(\forall \ell \in \mathbb{N})(f(i,j) \neq g(k,\ell)).$$

4. (20 pts.) Grade these answers.

You be the grader. Students have submitted the following answers to several exam questions. Assign each student answer either an A (correct yes/no answer, valid justification), a D (correct yes/no answer, invalid justification), or an F (incorrect answer). As always, $\pi = 3.14159\ldots$

1. **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
   Student answer: Yes. $2\pi = 6.283\ldots$, which is less than 100. Also $\pi = 3.1459\ldots$ is less than 50. Therefore the proposition is of the form True $\implies$ True, which is true.
2. **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
   Student answer: Yes. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$.
3. **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 49$. Explain your answer.
   Student answer: No. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$, which does not imply $\pi < 49$.
4. **Exam question:** Is the following proposition true? $\pi^2 < 5 \implies \pi < 5$. Explain your answer.
   Student answer: No, it is false. $\pi^2 = 9.87\ldots$, which is not less than 5, so the premise is false. You can't start from a faulty premise.

5. (20 pts.) How many different logical functions are there?

Given two propositions p and q, we have shown how to define functions of them (like $p \wedge q$ and $p \vee q$) by writing down a truth table, with one row per possible pair of values of (p,q), and one column showing the value of the expression (T or F) for every possible input (p,q). Indeed, there is a one-to-one correspondence between every possible truth table and every possible different logical function of p and q.

Part 1. How many different logical functions (i.e. different truth tables) are there of 2 propositions p and q?

Part 2. One can also have logical functions of more than 2 propositions, for example $(p_1 \wedge \neg p_2) \vee (\neg p_3)$ is a function of three propositions, p_1, p_2 and p_3. Again, each such function can be described by a unique truth table. Consider a function of n propositions $p_1, \ldots, p_n$. How many rows (as a function of n) would its truth table have? How many different functions (truth tables) are there for n propositions?

Part 3. Which do you think is larger, the number of truth tables for all functions of 9 propositions, or the number of atoms in the observable universe?

6. (20 pts.) Expressing an arbitrary logical function.

In the last question we learned to think about a logical function as a truth table. In this question we will see how to take an arbitrary truth table and write down a logical expression, using $\neg$, $\wedge$, $\vee$, and parentheses, that implements the function.

Part 1. Let's start with a truth table for just two propositions, p_1 and p_2. For each row of the truth table, write down an expression (using $\neg$ and $\wedge$) that is true only for that row of the truth table.

Part 2. Given an arbitrary truth table, explain how to combine the expressions from Part 1 to match the truth table. Illustrate this for the function that is false only when p_1 is true and p_2 is false.

Part 3. Generalize Parts 1 and 2 to a logical function of n propositions $p_1,...,p_n$.