CS 70 Discrete Mathematics and Probability Theory
Spring 2011 Demmel HW13

Due Apr 29, 8:00am

You may work in groups of up to 3 people (no larger!). Please read the group collaboration policies on bSpace or `www.cs.berkeley.edu/~demmel/cs70_Spr11` before beginning group work. You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (from any semester, not even a draft), nor share your own solutions (not even a draft).

Please begin your answer to each question on a new sheet of paper, and make sure that each sheet is labeled with your name, GSI name, the assignment number, the question number, and "CS70-Spring 2011".

Turn in each question in a different box in 283 Soda Hall: Question 1 in the box labeled "CS70 - 1", Question 2 in the box labeled "CS70 - 2", etc. Reason: Different problems will be graded in parallel by different readers.

Warning: You risk receiving no credit, or losing points, for any homework that does not conform to the above regulations! Please take the time to write clear and concise solutions; we will not grade messy or unreadable submissions. No late homeworks will be accepted. We will drop the lowest two homework scores.

1. (20 pts.) Lunch Date

Alice and Bob agree to try to meet for lunch between 12 and 1pm at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within that hour. In order to avoid wasting precious time, if the other person is not there when they arrive, they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch? (Hint: draw a picture.)

2. (20 pts.) Laplace distribution

A continuous random variable X is said to have a Laplace distribution with parameter λ if its pdf is given by:

$$f(x) = A\exp\left(-\lambda|x|\right), \qquad -\infty < x < \infty$$

for some constant A.

(a) Can the parameter λ be negative? Can λ be zero? Explain.

(b) Compute the constant A in terms of λ. Sketch the pdf.

(c) Compute the mean and variance of X in terms of λ.

(d) Compute $\Pr[X \geq x]$ in terms of x and λ. (Note: x can be positive or negative. Consider all cases.)

(e) For $s,t > 0$, compute $\Pr[X \geq s+t | X \geq s]$.

(f) Let $Y = |X|$. Compute the pdf of Y.

3. (20 pts.) Bus waiting times

You show up at a bus stop at a random time. There are three buses running on a periodic schedule. They all go to the same destination (which is where you want to go). Each of the three buses arrives at your bus stop once every 10 minutes, but at different offsets from each other. The offsets of the three buses are uniformly random and independent. You will get on the first bus to arrive at your bus stop. Let the random variable T denote the number of minutes you have to wait until the first bus arrives.

1. Compute an expression for the probability density function (pdf) for T.
2. Plot the pdf.
3. Using your pdf, compute $\mathbb{E}(T)$.

4. (20 pts.) z-scores

Let Z be a random variable (r.v.) that has a standard normal distribution, i.e., it is normally distributed with mean 0 and variance 1. Given a real number z, there are tables that allow us to compute $\Pr[Z \leq z]$ as a function of z. Such tables were very useful in the days before laptop computers and easy numeric integration. The value z is sometimes called a z-score. The tables allow us to compute one (or both) of the following quantities:

- The "left tail": The left tail represents the values of Z that are less than or equal to z, and $\Pr[Z \leq z]$ is the area under the normal curve and where $x \leq z$. For instance, $\Pr[Z \leq -1] \approx 0.1587$, $\Pr[Z \leq 0] = 0.5$, and $\Pr[Z \leq 1] \approx 0.8413$.
- The "right tail": The right tail represents the values greater than or equal to z. For instance, $\Pr[Z \geq 1] \approx 0.1587$, and $\Pr[Z \geq 2] \approx 0.0228$.

You can find resources for calculating these values—a normal table and a normal calculator—at `http://math2.org/math/stat/distributions/z-dist.htm`. These allow you to go back and forth between a z-score z and the probability $\Pr[Z \leq z]$ (i.e., the area under the "left tail"), or between z and $\Pr[Z \geq z]$ (i.e., the area under the "right tail"). Also see the table in discussion section 13.

1. Let the r.v. Z be normally distributed with mean 0 and variance 1. Use a table or calculator mentioned above to find the approximate value of $\Pr[Z \leq 1.5]$ and $\Pr[|Z| \leq 1.5]$.

z-scores have many applications. For instance, if the random variable X is normally distributed with mean μ and variance σ^2, then the random variable X can be *normalized* to get a random variable Z defined by $Z = (X - \mu)/\sigma$. A useful fact is that, under these assumptions, Z will be normally distributed with mean 0 and variance 1. Note that the value of Z can be viewed as a z-score. Although in the old days, this was viewed as merely a way to allow the use of standard normal tables, thinking about the appropriate normalizations is useful even today.

2. Suppose X is normally distributed with mean 100 and standard deviation 10. Calculate $\Pr[X \geq 125]$. You may wish to use the resources listed above.
3. Suppose X is a r.v. with a normal distribution and mean $\mu = 20$ and variance $\sigma^2 = 25$. Calculate $\Pr[X \leq 25]$ and $\Pr[15 \leq X \leq 25]$.

5. (20 pts.) z-scores, continued

1. DigiPart sells 100 Ω resistors. However, when you buy a resistor from DigiPart, its resistance is not actually guaranteed to be precisely 100 Ω. Rather, the resistance is normally distributed with mean 100 Ω and standard deviation 2 Ω. If you buy a 100 Ω resistor from DigiPart, what is the probability that its resistance is between 96 Ω and 104 Ω?

2. The height of American adult females is approximately normally distributed, with an average of 64 inches (5' 4") and a standard deviation of 2.7 inches. Approximately what fraction of American adult females are at least 68 inches tall? If we form a basketball team by picking 5 American adult females at random, calculate the probability that at least one of them is at least 6 feet (72 inches) tall.

6. (20 pts.) The normal approximation to the binomial

Suppose $B \sim \text{Binomial}(n, p)$, i.e., the r.v. B is binomially distributed: it is the number of heads after flipping n coins, with heads probability p. We have seen previously $\mathbb{E}(B) = np$ and $\text{Var}(B) = np(1-p)$. It turns out that, for large n, the binomial distribution B approximates the normal distribution with the same mean and variance. A standard rule of thumb is that the normal approximation is a reasonable approximation if $np \geq 5$ and $n(1-p) \geq 5$. Use this fact to solve the following questions.

1. Suppose that the final exam for an inferior Discrete Math class with a lazy prof (not at UC!) has 80 multiple-choice questions, where each question has 4 choices. If you guess blindly, you have a $1/4$ chance of guessing right on each question. Calculate the approximate probability that, if you answer every question by guessing blindly, you get 30 or more questions right.

   Hint: Approximate the number of right answers as a normal distribution, normalize it to obtain a standard normal distribution, then use a normal table, such as in question 4.

2. Find a value k for which, when you flip a fair coin 10,000 times, the probability of k or more heads is approximately 0.20.

   Hint: Approximate the number of heads as a r.v. X with an appropriate normal distribution, normalize it to obtain a standard normal distribution, then use a normal table, as above.