CS 70 Discrete Mathematics and Probability Theory
Spring 2011 Demmel HW 2

Due Feb 3, 6:30pm

Please follow the directions on the first homework (and class web page) about labeling and turning in your homework, and about collaboration.

1. (20 pts.) Useful properties of integers.
Two positive integers m and n are called *relatively prime* if the largest integer d that divides both of them evenly is $d = 1$.

Part 1. Let n be any positive integer. Prove that n and $n+1$ are relatively prime.

Part 2. Let m and n be any two positive integers. Prove that there are other positive integers d, x and y with the following properties:

- $m = d \cdot x$
- $n = d \cdot y$
- x and y are relatively prime

Use induction to do your proof. Hint: do induction on the value of $\max(m,n)$. Note: d is called the *greatest common divisor* of m and n.

2. (35 pts.) Practice with proofs.
Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Lecture Note 2) you used. Recall that a real number x is *rational* if and only if it can be written as a quotient $x = m/n$ where m and n are integers, and $n \neq 0$; otherwise x is *irrational*.

1. For all natural numbers n, if n is even then $n^2 + 2011$ is odd.
2. For all integers n, $n^{300} - 2n^2 - 7n + 1$ is odd.
3. For all real numbers a and b, if $a + b \geq 2011$ then either $a \geq -1776$ or $b \geq 3787$.
4. For all real numbers a and b, if a and $a + b$ are irrational, then b is irrational.
5. For all natural numbers n, $100n^3 > n!$.

3. (20 pts.) Practice with induction proofs.
Prove each of the following statements by using induction on n.

1. For $n \geq 2$, let $s_n = (1 - \frac{1}{2^2}) \cdot (1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{n^2})$. Prove that $s_n = .5(1 + \frac{1}{n})$.
2. For $n \geq 1$, let $t_n = \sum_{k=1}^{n} k \cdot 2^k$. Prove that $t_n = (n-1)2^{n+1} + 2$.

4. (14 pts.) Grade these answers
You be the grader. Students have submitted the following proofs. Decide whether you think the proof is valid or not, and assign each student answer either an A (valid proof) or an F (invalid proof). If the proof

is invalid, explain *clearly and concisely* where the logical error in the proof is, including exactly which step of the reasoning is erroneous. (If you think the proof is correct, you do not need to give any explanation.) Simply saying that the claim is false is *not* an acceptable explanation.

1. **Claim:** For all natural numbers n, if $2n+2$ is a multiple of 4, then n^2+1 is a multiple of 4.

   Proof: We'll use a proof by contrapositive. Assume $2n+2$ is not a multiple of 4. There are four cases, depending upon the value of n:

   - **Case 0:** $n = 4k$, for some natural number k. In this case, $n^2+1 = (4k)^2+1 = 16k^2+1$, which is not a multiple of 4.
   - **Case 1:** $n = 4k+1$, for some natural number k. In this case, $n^2+1 = (4k+1)^2+1 = 16k^2+8k+2$, which is not a multiple of 4.
   - **Case 2:** $n = 4k+2$, for some natural number k. In this case, $n^2+1 = (4k+2)^2+1 = 16k^2+16k+5$, which is not a multiple of 4.
   - **Case 3:** $n = 4k+3$, for some natural number k. In this case, $n^2+1 = (4k+3)^2+1 = 16k^2+24k+10$, which is not a multiple of 4.

   In each of these cases, n^2+1 is not a multiple of 4. These four cases exhaust all possibilities, so it follows that n^2+1 is not a multiple of 4. This concludes the proof. □

2. **Claim:** For every real number x, if x is irrational, then $2011x$ is irrational.

   Proof: We will use proof by contrapositive. Suppose $2011x$ is rational. Then there exist integers p,q with $q \neq 0$ and $2011x = p/q$. Therefore $x = p/r$ where $r = 2011q$. Since p,r are integers with $r \neq 0$, this means that x is rational, too. Therefore, if $2011x$ is rational, then x is rational. By the contrapositive, it follows that if x is irrational, then $2011x$ is irrational. □

5. (18 pts.) Check for truth value

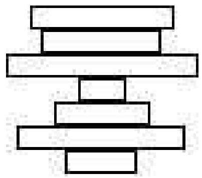
Let $P(n)$ denote the claim that $1+2+\cdots+n \leq 2011\,n$, $Q(n)$ denote the claim that $1+2+\cdots+n \leq (n^2-2)/2$, and $R(n)$ denote the claim that $1+2+\cdots+n \leq (n+1)^2/2$. For each proposition in parts 1–6 below, say whether the proposition is true or false. In parts 4–6, prove your answer (you do not need to prove your answer to parts 1–3).

1. $(\forall n \in \mathbb{N})(P(n))$.
2. $(\forall n \in \mathbb{N})(Q(n))$.
3. $(\forall n \in \mathbb{N})(R(n))$.
4. $(\forall n \in \mathbb{N})(P(n) \implies P(n+1))$.
5. $(\forall n \in \mathbb{N})(Q(n) \implies Q(n+1))$.
6. $(\forall n \in \mathbb{N})(R(n) \implies R(n+1))$.

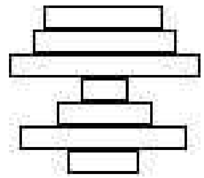
6. (20 pts.) The proof of the pi is in the eating

Dave is moonlighting as an intern at La Val's Pizza, where he is learning how to make pizzas. However, the delightful smell of the cooking pizzas tends to make him a bit distracted.

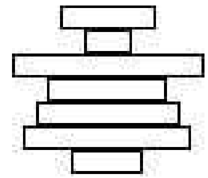
One day, he has a stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size, with the largest pizza on the bottom, the next largest pizza just above that, and so on. During his internship so far, he has learned how to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). The figure below shows two sample flips.



initial stack



after flipping top two pizzas in initial stack



after flipping top five pizzas in initial stack

This is the only move Dave can do to change the order of the stack; however, he is willing to keep repeating this kind of move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some sequence of moves, no matter how many pizzas he starts with and no matter how they are arranged initially? Prove your answer.