CS 70 Discrete Mathematics and Probability Theory
Spring 2011 Demmel HW 3

Due Feb 10, 6:00pm

You may work in groups of up to 3 people (no larger!). Please read the group collaboration policies on bSpace or `www.cs.berkeley.edu/~demmel/cs70_Spr11` before beginning group work. You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please begin your answer to each question on a new sheet of paper, and make sure that each sheet is labeled with your name, section number, GSI name, the assignment number, the question number, and "CS70–Spring 2011".

Turn in each question in a different box in 283 Soda Hall: Question 1 in the box labeled "CS70 - 1", Question 2 in the box labeled "CS70 - 2", etc. Reason: Different problems will be graded in parallel by different readers.

Warning: You risk receiving no credit, or losing points, for any homework that does not conform to the above regulations! Please take the time to write clear and concise solutions; we will not grade messy or unreadable submissions. No late homeworks will be accepted. We will drop the lowest two homework scores.

1. (10 pts.) Strengthening the proposition

For this problem, define

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

Now do the following parts.

1. Alice wants to prove the following statement, using simple induction on n: for all natural numbers $n > 1$, $s_n < 2$. Why is this difficult?
2. Prove the following statement, using simple induction on n: for all natural numbers $n > 1$, $s_n < 2 - \frac{1}{n}$.
3. Given your answer to part 2, now prove the following statement: for all natural numbers $n > 1$, $s_n < 2$. (**Hint:** Don't work too hard!)

Comment: This problem illustrates how sometimes the easiest way to prove a statement (e.g., $s_n < 2$) may actually be to prove a harder-looking statement (e.g., $s_n < 2 - \frac{1}{n}$).

2. (12 pts.) Principle of induction

Let $P(k)$ be a proposition involving a natural number k. Suppose you know only that $(\forall k \in \mathbb{N})(P(k) \implies P(k+2))$ is true. For each of the following propositions, say whether the proposition is (i) definitely true, (ii) definitely false, or (iii) possibly (but not necessarily) true. Give a *brief* (one or two sentences) explanation for each of your answers.

1. $(\forall n \in \mathbb{N})(P(n))$.
2. $(\forall n \in \mathbb{N})(\neg P(n))$.
3. $P(0) \implies (\forall n \in \mathbb{N})(P(n+2))$.
4. $(P(0) \wedge P(1)) \implies (\forall n \in \mathbb{N})(P(n))$.
5. $P(n) \implies ((\exists m \in \mathbb{N})(m > n+2010 \wedge P(m)))$.
6. $(\forall n \in \mathbb{N})(n < 2010 \implies P(n)) \wedge (\forall n \in \mathbb{N})(n \geq 2010 \implies \neg P(n))$.

3. (10 pts.) Linear recurrence relations

Give an explicit expression for the solution of

$$F(n) = -F(n-1) + 6 * F(n-2)$$

$$F(0) = F(1) = 1$$

as a function of n, and prove it is correct.

4. (10 pts.) Marbles

There is a bucket that contains some number of gold-colored marbles, silver-colored marbles, and bronze-colored marbles. When it is a player's turn, the player may either: (i) remove 1 gold marble from the bucket, and add up to 4 silver marbles into the bucket; (ii) remove 3 silver marbles from the bucket, and add up to 8 bronze marbles into the bucket; or, (iii) remove 1 bronze marble from the bucket. These are the only legal moves. The last player that can make a legal move wins.

Prove by induction that, if the bucket initially contains a finite number of marbles at the start of the game, then the game will end after a finite number of moves.

5. (10 pts.) Let's be social

n people go to a bar. Initially, each person sits at their own table. After a little while, the bartender picks a table, taps the person at that table on the shoulder, and asks him to move to a second table. The person who just moved introduces himself and shakes hands with the person who was already sitting at the second table.

In general, the bartender keeps repeating the following operation: the bartender chooses two tables; the bartender asks everyone sitting at the first table to move over to the second table; and each of the folks who just moved from the first table shake hands with everyone who was already sitting at the second table. Suppose that there were k people sitting at the first table and ℓ people sitting at the second table before this operation. After this operation, there are 0 people at the first table and $k+\ell$ people at the new table. Also, each of the k newcomers shakes hands with each of the ℓ folks already at the second table, so $k\ell$ handshakes occur during this operation. The bartender repeats this kind of operation until all n people are sitting at the same table.

Let $H(n)$ denote the the total number of handshakes that have occurred among the n people by the time this process is finished and everyone is seated at the same table. Prove that it doesn't matter what order the bartender decides to choose tables; we always have $H(n) = n(n-1)/2$.

Hint: Use strong induction.

6. (10 pts.) Stable marriage

1. Run the traditional propose and reject algorithm on the following example.
   Men's preference list:

1	*A*	*B*	*C*	*D*
2	*B*	*C*	*A*	*D*
3	*C*	*A*	*B*	*D*
4	*A*	*B*	*C*	*D*

   Women's preference list:

A	2	3	4	1
B	3	4	1	2
C	4	1	2	3
D	1	2	3	4

   Also, for this stable marriage instance, give an example of an *unstable* pairing and explain why it is unstable.
2. For each of the following claims, state whether the claim is true or false. If it is true, give a *short* proof; if it is false, give a *simple* counterexample.
   (a) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.
   (b) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.