MATH 110: LINEAR ALGEBRA
HOMEWORK #3

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§2.1: Linear Transformations, Null Spaces, and Ranges

Problem 1. Here V and W are vector spaces over a field $\mathbb{F}$ and $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ (but T may not be linear).

(a) True. I can't find it defined in the book, but "T preserves sums and scalar products" is just a short way of saying that T satisfies conditions (a) and (b) in the definition of linear transformation.

(b) False. (Note firstly that the statement is to be read "if $\forall x, y\, [\mathrm{T}(x+y) = \mathrm{T}(x) + \mathrm{T}(y)]$ then T is linear".) If the base field $\mathbb{F} = \mathbb{Z}_p$ then the statement is true. (To see T preserves scalar multiplication, let $c \in \mathbb{F}$. Then there is some $n < p$ such that $c = 1 + \ldots + 1$, where there are n terms on the right side. Then use distribution and preservation of addition to get $\mathrm{T}(cv) = c\mathrm{T}(v)$.) One can argue similarly to show the statement is also true if $\mathbb{F} = \mathbb{Q}$.
To get a counter-example, we need $\mathbb{F}$ to be less trivial. For some motivation, consider Theorem 2.1. It tells us that if $\mathbf{Rg}(\mathrm{T})$ is not a subspace of W, then T is not linear. So we can look for T like this. Let $\mathrm{V} = \mathrm{W} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{F} = \mathbb{Q}(\sqrt{2})$. (Here we use the fact that any field is a vector space over itself.) Note that the only subspaces of W (or V) are $\{0\}$ and $\mathbb{Q}(\sqrt{2})$ (why is $\mathbb{Q}$ not a subspace?). So T will be non-linear if its range is between these two sets - i.e. $0 \subsetneq \mathbf{Rg}(\mathrm{T}) \subsetneq \mathbb{Q}(\sqrt{2})$. But we can get $\mathbf{Rg}(\mathrm{T}) = \mathbb{Q}$ defining T by:

$$\mathrm{T}(q + r\sqrt{2}) = q$$

where $q, r \in \mathbb{Q}$. There are a few things to check.
Firstly we need to know that this really defines a function. That is, given $v \in \mathrm{V}$, there must be exactly one value that we're telling T to send v to. If we had $v = q_1 + r_1\sqrt{2} = q_2 + r_2\sqrt{2}$ (where the q's and r's are in $\mathbb{Q}$), but $q_1 \neq q_2$, then we'd be giving conflicting instructions to T: both $\mathrm{T}(v) = q_1$ and $\mathrm{T}(v) = q_2$. But considering $\mathbb{Q}(\sqrt{2})$ as a vector space over $\mathbb{Q}$, the set $\{1, \sqrt{2}\}$ is a basis (linear independence comes down to the fact that $\sqrt{2} \notin \mathbb{Q}$). So if $v = q_1 + r_1\sqrt{2} = q_2 + r_2\sqrt{2}$, then by uniqueness of representation of v as a linear combination of the basis, $q_1 = q_2$ and $r_1 = r_2$. So there is no such problem. Also, we have defined T on all elements of V, as every $v \in \mathrm{V}$ is of the form $q + r\sqrt{2}$ as above. Thus T is a function.
It's easy to check T preserves addition.
It's also easy to check that $\mathbf{Rg}(\mathrm{T}) = \mathbb{Q}$. So T cannot be linear, as discussed above. Thus T does not preserve scalar multiplication. However, we can check this directly. T will preserve scalar multiplication for scalars in $\mathbb{Q}$, so we should check $\sqrt{2}$. Note

Date: September 22.

that $T(1) = T(1 + 0\sqrt{2}) = 1$, so $\sqrt{2}T(1) = \sqrt{2}$. But $T(\sqrt{2}.1) = T(0 + 1.\sqrt{2}) = 0$. Thus T does not preserve scalar multiplication.
Note also that the nullspace of T is not a subspace of V.
Verifying that T is a function can also be done by appealing to Theorem 2.6 (the argument above is similar to that proof). For we can let $V' = W' = \mathbb{Q}(\sqrt{2})$ be vector spaces over $\mathbb{Q}$, so $\{1, \sqrt{2}\}$ is a basis for V'. By Theorem 2.6, there is a unique linear $T' : V' \to W'$ such that $T(1) = 1$ and $T(\sqrt{2}) = 0$. Then by linearity, T' satisfies the equation given above for T. This is not a contradiction as T' is only guaranteed to be linear in terms of V' and W' - i.e., with field of scalars $\mathbb{Q}$. When we extend the field of scalars to $\mathbb{Q}(\sqrt{2})$, scalar multiplication is no longer preserved.

(c) False. E.g. $V = W = \mathbb{R}$, $T(0) = 0$, $T(x) = 1$ if $x \neq 0$. (If T is linear it is true.)
(d) True. $T(0_V) = T(0_{\mathbb{F}} 0_V) = 0_{\mathbb{F}} T(0_V) = 0_W$, using linearity for the second equality.
(e) False. If $\dim(V) \neq \dim(W)$, then by the dimension theorem, $\text{nullity}(T) + \text{rank}(T) = \dim(V) \neq \dim(W)$.
(f) False. E.g. The 0-map, where $\dim(V) > 0$.
(g) True. Corollary to Theorem 2.6.
(h) False. E.g. if x_1, x_2 are dependent but y_1, y_2 are independent.

For a few of the following problems, I'll use the following simple fact, which sits naturally with Theorems 2.4 and 2.5:

Fact 1. Suppose V and W are finite dimensional vector spaces over field a $\mathbb{F}$ and $T : V \to W$ is linear. Then T is 1-1 iff $\text{nullity}(T) = 0$ and T is onto iff $\text{rank}(T) = \dim(W)$.

Proof. The first statement is simply a rephrasing of Theorem 2.4, as $\text{nullity}(T) = 0 \iff \mathbf{N}(T) = \{0\}$.
For the second, if T is onto then $\mathbf{Rg}(T) = W$, so $\text{rank}(T) = \dim(\mathbf{Rg}(T)) = \dim(W)$. On the other hand, $\mathbf{Rg}(T)$ is a subspace of W, so if $\dim(\mathbf{Rg}(T)) = \dim(W)$ then by Theorem 1.11, we must have $\mathbf{Rg}(T) = W$, i.e., T is onto. □

Problem 3. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

T is linear.

$$T(a + b) = T(a_1 + b_1, a_2 + b_2) = ((a_1 + b_1) + (a_2 + b_2), 0, 2(a_1 + b_1) - (a_2 + b_2)) =$$
$$= ((a_1 + a_2) + (b_1 + b_2), 0, (2a_1 - a_2) + (2b_1 - b_2)) = T(a) + T(b).$$

$$T(ra) = T(ra_1, ra_2, ra_3) = (ra_1 + ra_2, 0, 2ra_1 - ra_2) =$$
$$(r(a_1 + a_2), 0, r(2a_1 - a_2)) = rT(a).$$

Nullspace.

$$a \in \mathbf{N}(T) \iff (a_1 + a_2, 0, 2a_1 - a_2) = 0 \iff$$
$$\iff a_1 = -a_2 \;\&\; a_1 = a_2/2 \iff a_1 = a_2 = 0 \iff a = 0.$$

So $\mathbf{N}(T) = \{0\}$. Therefore $\emptyset$ is a (the) basis for $\mathbf{N}(T)$.

Range.

$$\mathbf{Rg}(T) = \{T(a) | a \in \mathbb{R}^2\} = \{(a_1 + a_2, 0, 2a_1 - a_2) | (a_1, a_2) \in \mathbb{R}^2\} =$$

$$= \{a_1(1,0,2) + a_2(1,0,-1) | a_1, a_2 \in \mathbb{R}\} = \text{span}((1,0,2),(1,0,-1)).$$

The last equality is by definition of span: notice that the set on the left is the set of all linear combinations of the vectors $(1,0,2)$ and $(1,0,-1)$. The set $\mathcal{B} = \{(1,0,2),(1,0,-1)\}$ is independent (the computation is done in showing $\mathbf{N}(\mathrm{T}) = \emptyset$). Since $\mathcal{B}$ also spans $\mathbf{Rg}(\mathrm{T})$, it is a basis. (If we didn't have to verify the dimension theorem, we could actually apply the dimension theorem here to conclude that it is independent without checking it, for the dimension theorem tells us that there must be 2 vectors in a basis for $\mathbf{Rg}(\mathrm{T})$.)

Dimension Theorem.
From the above, $\text{nullity}(\mathrm{T}) = 0$ and $\text{rank}(\mathrm{T}) = 2$, and $0 + 2 = 2 = \dim(\mathbb{R}^2)$, agreeing with the dimension theorem.

1-1/onto.
As $\text{nullity}(\mathrm{T}) = 0$ and $\text{rank}(\mathrm{T}) = 2 < \dim(\mathrm{W}) = 3$, the fact following problem 1 tells us that T is 1-1 but not onto.

Problem 4. Let $\mathrm{V} = \mathrm{M}_{2*3}(\mathbb{F})$, $\mathrm{W} = \mathrm{M}_{2*2}(\mathbb{F})$.

T is linear.
I will use the criterion specified in remark 2 following the definition of linearity in the textbook. Let $A, B \in \mathrm{V}$ and $c \in \mathbb{F}$. Then $\mathrm{T}(A + cB) =$

$$= \mathrm{T}\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} cB_{11} & cB_{12} & cB_{13} \\ cB_{21} & cB_{22} & cB_{23} \end{bmatrix}\right)$$

$$= \mathrm{T}\left(\begin{bmatrix} A_{11} + cB_{11} & A_{12} + cB_{12} & A_{13} + cB_{13} \\ A_{21} + cB_{21} & A_{22} + cB_{22} & A_{23} + cB_{23} \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(A_{11} + cB_{11}) - (A_{12} + cB_{12}) & A_{13} + cB_{13} + 2(A_{12} + cB_{12}) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (2A_{11} - A_{12}) + (2cB_{11} - cB_{12}) & (A_{13} + 2A_{12}) + (cB_{13} + 2cB_{12}) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (2A_{11} - A_{12}) + c(2B_{11} - B_{12}) & (A_{13} + 2A_{12}) + c(B_{13} + 2B_{12}) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2A_{11} - A_{12} & A_{13} + 2A_{12} \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} (2B_{11} - B_{12}) & (B_{13} + 2B_{12}) \\ 0 & 0 \end{bmatrix} = \mathrm{T}(A) + c\mathrm{T}(B).$$

Nullspace.

$$\mathrm{T}(A) = 0 \iff \begin{bmatrix} 2A_{11} - A_{12} & A_{13} + 2A_{12} \\ 0 & 0 \end{bmatrix} = 0 \iff A_{11} = A_{12}/2 \ \&\ A_{13} = -2A_{12},$$

where I'm assuming $\text{char}(\mathbb{F}) \neq 2$. So

$$\mathbf{N}(\mathrm{T}) = \{A \in \mathrm{V} | A_{11} = A_{12}/2 \ \&\ A_{13} = -2A_{12}\}$$

$$= \left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} | a_{ij} \in \mathbb{F},\ a_{11} = a_{12}/2 \ \&\ a_{13} = -2a_{12}\right\}$$

$$= \left\{\begin{bmatrix} a/2 & a & -2a \\ b_1 & b_2 & b_3 \end{bmatrix} | a, b_i \in \mathbb{F}\right\}$$

$$= \left\{ aF + b_1 E^{(21)} + b_2 E^{(22)} + b_3 E^{(23)} \, | a, b_i \in \mathbb{F} \right\}, \tag{1}$$

where

$$F = \begin{bmatrix} 1/2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

and the $E^{(kl)}$ are the standard basis matrices for V (so $E^{(kl)}$ has a 1 in the k^{th} row, l^{th} column, and 0 elsewhere).
So we have that $\mathbf{N}(\mathrm{T}) = \operatorname{span}(\mathcal{B})$, where $\mathcal{B} = \{F, E^{(21)}, E^{(22)}, E^{(23)}\}$ (by the form of $\mathbf{N}(\mathrm{T})$ in (1), similarly to the range in the previous problem). It's also clear that $\mathcal{B}$ is linearly independent, so it is a basis for $\mathbf{N}(\mathrm{T})$.

Range.
Let

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Then $\mathbf{Rg}(\mathrm{T}) = \operatorname{span}(\mathcal{B}')$. To see this, we check that each side is a subset of the other. From the definition of T, we clearly have $\mathbf{Rg}(\mathrm{T}) \subseteq \operatorname{span}(\mathcal{B}')$. To see $\supseteq$, note that

$$\mathrm{T}\left(\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and similarly, we can get the second matrix in $\mathcal{B}'$ to be hit by T (meaning in $\mathbf{Rg}(\mathrm{T})$). So $\mathcal{B}' \subseteq \mathbf{Rg}(\mathrm{T})$. But $\mathbf{Rg}(\mathrm{T})$ is a subspace of W, so by Theorem 1.5, $\operatorname{span}(\mathcal{B}') \subseteq \mathbf{Rg}(\mathrm{T})$. As $\mathcal{B}$ is also independent, it is a basis for $\mathbf{Rg}(\mathrm{T})$.

Dimensions.
Inspecting the bases we found, $\mathcal{B}$ and $\mathcal{B}'$, for the nullspace and range respectively, we have $\operatorname{nullity}(\mathrm{T}) = 4$ and $\operatorname{rank}(\mathrm{T}) = 2$. $\dim(\mathrm{V}) = 6$.

1-1/onto.
By Theorem 2.4, T is not 1-1 (it's nullspace is non-trivial). $\operatorname{rank}(\mathrm{T}) = 4 < \dim(\mathrm{W})$, so by the fact following problem 1, T is not onto.

Problem 5. Let $\mathrm{V} = \mathrm{P}_2(\mathbb{R})$ and $\mathrm{W} = \mathrm{P}_3(\mathbb{R})$.

T is linear.
Let $f, g \in \mathrm{V}$ and $c \in \mathbb{R}$. Then

$$\mathrm{T}(f+cg) = x(f+cg)+(f+cg)' = xf+xcg+f'+cg' = (xf+f')+c(xg+g') = \mathrm{T}(f)+c\mathrm{T}(g).$$

Note that the linearity of the derivative has been used to obtain the second equality.

Nullspace.
I will use the standard basis to represent functions in $\mathrm{P}_2(\mathbb{R})$. That basis is $\{x^2, x, \mathbf{1}\}$. Here $x \notin \mathbb{R}$, but $x \in \mathrm{V} = \mathrm{P}_2(\mathbb{R})$, i.e. x is a quadratic function - the identity function, defined by $x(r) = r$ for $r \in \mathbb{R}$. x^2 denotes the square function - for $r \in \mathbb{R}$, $x^2(r) = r^2$. $\mathbf{1}$ denotes the constant function taking value 1. Also, $x^3 \in \mathrm{W} = \mathrm{P}_3(\mathbb{R})$ will denote the cube function.
For $f \in \mathrm{V}$, let $f_2, f_1, f_0 \in \mathbb{R}$ be f's coefficients with respect to this basis, so $f = f_2 x^2 + f_1 x + f_0$ (it should read $f_0\mathbf{1}$ but one can identify some $a \in \mathbb{R}$ with the constant function $a\mathbf{1}$, which

I'll do).
A digression: note that in the definition of T, $\mathrm{T}(f) = xf + f'$, we are multiplying two vectors, x and f, and producing a vector $xf \in \mathrm{W}$. In some situations, such as this, we can make sense of vector multiplication (but note that V isn't closed under this multiplication).
Anyway, the nullspace:

$$\mathrm{T}(f) = 0 \iff xf + f' = 0 \iff x(f_2x^2 + f_1x + f_0) + 2f_2x + f_1 = 0 \iff$$
$$\iff f_2x^3 + f_1x^2 + (f_0 + 2f_2)x + f_1 = 0 \iff f_2 = f_1 = (f_0 + 2f_2) = 0 \iff$$
$$\iff f_2 = f_1 = f_0 = 0 \iff f = 0.$$

Note we have used the fact that $\{x^3, x^2, x, \mathbf{1}\}$ is a linearly independent subset of W, to obtain the equivalence of the 4^{th} and 5^{th} statements. Therefore $\mathbf{N}(\mathrm{T}) = \{0\}$, so $\emptyset$ is the basis for the nullspace.

Range.
Using some of the calculations just done,

$$\mathbf{Rg}(\mathrm{T}) = \{\mathrm{T}(f)|f \in \mathrm{V}\} = \{ax^3 + bx^2 + (c + 2a)x + b|a, b, c \in \mathbb{R}\} =$$
$$= \{a(x^3 + 2x) + b(x^2 + 1) + cx|a, b, c \in \mathbb{R}\} =$$
$$= \mathrm{span}(x^3 + 2x, x^2 + 1, x) = \mathrm{span}(x^3, x^2 + 1, x).$$

Note that the second to last equality is by definition of span. For the last equality it is enough to see that x^3 is in $\mathrm{span}(x^3 + 2x, x^2 + 1, x)$ and that $x^3 + 2x$ is in $\mathrm{span}(x^3, x^2 + 1, x)$. Finally, the set $\{x^3, x^2 + 1, x\}$ is clearly linearly independent (remember the 0 polynomial must have all coefficients 0). As this set also spans the range, it is a basis for the range. (Again, we could use the dimension theorem here to conclude linear independence of the set.)

Dimension Theorem.
From the bases given, we have that $\mathrm{rank}(\mathrm{T}) = 3$ and $\mathrm{nullity}(\mathrm{T}) = 0$. Verifyingly, $\dim(\mathrm{P}_2(\mathbb{R})) = 3$.

1-1/onto.
$\mathrm{nullity}(\mathrm{T}) = 0$ and $\dim(\mathrm{P}_3(\mathbb{R})) = 4 > \mathrm{rank}(\mathrm{T})$, so by the fact following problem 1, T is 1-1, but not onto.

Problem 6. Let $\mathrm{V} = \mathrm{M}_{n*n}(\mathbb{F})$ and $\mathrm{W} = \mathbb{F}$.
If you couldn't figure this problem out, before reading the solution (if you're about to), you should try doing it in the $n = 2$ case, as it is simpler, then the $n = 3$ case, which has one more idea. The general case is really no different to the latter. Also, it's easy to see what rank(tr) is, so using the dimension theorem, this can give you a hint about the nullspace. So stop reading now.

Linearity.
Let $A, B \in \mathrm{V}$ and $c \in \mathbb{F}$. Then

$$\mathrm{tr}(A + cB) = \sum_{i=1}^{i=n}(A + cB)_{ii} = \sum_{i=1}^{i=n}(A_{ii} + cB_{ii}) = \sum_{i=1}^{i=n} A_{ii} + c\sum_{i=1}^{i=n} B_{ii} = \mathrm{tr}(A) + c\,\mathrm{tr}(B).$$

Nullspace.
This is the most difficult part. The point is that the condition $\mathrm{tr}(A) = 0$ is just one solvable linear equation in (some of) the entries in A, so we can allow all entries to vary freely except one on the diagonal, and then set that one to satisfy the equation. Another way to see something like this will happen, is to note first that rank(T) = 1 (see below), and as $\dim(\mathrm{V}) = n^2$, we must have $\mathrm{nullity}(\mathrm{T}) = n^2 - 1$ by the dimension theorem. So there will be $n^2 - 1$ elements in the basis for the nullspace, which is why we can allow all but one $(n^2 - 1)$ of the entries to vary freely. Other than that we just need good notation.

$$A \in \mathbf{N}(\mathrm{tr}) \iff \mathrm{tr}(A) = 0 \iff \sum_{i=1}^{i=n} A_{ii} = 0 \iff A_{nn} = -\sum_{i=1}^{i=n-1} A_{ii}$$

As we've reduced membership in $\mathbf{N}(\mathrm{tr})$ to $A_{nn} = -\sum_{i=1}^{i=n-1} A_{ii}$ (where the sum is 0 if $n = 1$), we have

$$\mathbf{N}(\mathrm{tr}) = \left\{ A \in \mathrm{V} \middle| A_{nn} = -\sum_{i=1}^{i=n-1} A_{ii} \right\}$$

$$= \left\{ \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \middle| a_{ij} \in \mathbb{F}, a_{nn} = -\sum_{i=1}^{i=n-1} a_{ii} \right\}$$

$$= \left\{ \begin{bmatrix} a_{11} & \dots & a_{1,n-1} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & \dots & a_{n,n-1} & -\sum_{i=1}^{i=n-1} a_{ii} \end{bmatrix} \middle| a_{ij} \in \mathbb{F} \right\}.$$

Now let $E^{(kl)}$, for $1 \leq k, l \leq n$, be the standard basis vectors for V. (So $E^{(kl)}$ has a 1 in the k^{th} row, l^{th} column, and 0 elsewhere.) For $1 \leq k \leq n-1$, let $D^{(k)} = E^{(kk)} - E^{(nn)}$. Then

$$\begin{bmatrix} a_{11} & \dots & a_{1,n-1} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & \dots & a_{n,n-1} & -\sum_{i=1}^{i=n-1} a_{ii} \end{bmatrix} = \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} a_{ij} E^{(ij)} + \sum_{i=1}^{i=n-1} a_{ii} D^{(i)}.$$

To see this equation holds, consider the $n = 2$ and $n = 3$ cases first. Write the matrix as a sum of matrices, one for each a_{ij} (where $i \neq n$ or $j \neq n$). Combining this equation and the description of the nullspace above, we have $\mathbf{N}(\mathrm{tr}) = \mathrm{span}(\mathcal{B})$, where

$$\mathcal{B} = \{E^{(kl)} | 1 \leq k, l \leq n, k \neq l\} \cup \{D^{(k)} | 1 \leq k \leq n-1\}).$$

It's easy to check directly that $\mathcal{B}$ is linearly independent. So we've constructed a basis for $\mathbf{N}(\mathrm{tr})$.

Range.
tr is linear, so its range must be a subspace of $\mathbb{F}$. $\mathbb{F}$ is 1-dimensional, so its only subspaces are $\{0\}$ and $\mathbb{F}$. Clearly $1 \in \mathbf{Rg}(\mathrm{tr})$, as $\mathrm{tr}(E^{(nn)}) = 1$, for example. So $\mathbf{Rg}(\mathrm{tr}) = \mathbb{F}$ and a basis for it is $\{1\}$.

Demented Theorem.
We had n^2-1 elements in our basis for $\mathbf{N}(\mathrm{tr})$, so $\mathrm{nullity}(\mathrm{tr}) = n^2-1$, and $\mathrm{rank}(\mathrm{tr}) = 1$, and $\dim(\mathrm{V}) = n^2$.
1-1/onto.
We already saw $\mathbf{Rg}(\mathrm{tr}) = \mathbb{F}$, so tr is onto. Clearly it's not 1-1?? Note quite - using that $\mathrm{nullity}(\mathrm{tr}) = n^2-1$ from above, and the fact following problem 1, if $n > 1$, then tr is not one-one, but if $n = 1$, then tr is one-one. (This makes sense as in the $n = 1$ case $\mathrm{tr}(A) = A_{11}$, the single entry of A.)

Problem 7. Let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ where V and W are vector spaces over the field $\mathbb{F}$. (T is not assumed to be linear.)

1. This is done in problem 1(d).

2. This property easily holds if T is linear. So let us assume the property holds, and show that T is linear. We know that

$$\forall x, y \in \mathrm{V}\ \forall c \in \mathbb{F}\ \ [\mathrm{T}(cx+y) = c\mathrm{T}(x) + \mathrm{T}(y)] \tag{2}$$

   So let $x, y \in \mathrm{V}$. We have $\mathrm{T}(x+y) = \mathrm{T}(1x+y) = 1\mathrm{T}(x)+\mathrm{T}(y) = \mathrm{T}(x)+\mathrm{T}(y)$ where the middle equality holds by (2). So T preserves sums. To see T preserves scalar multiplication, first note that $\mathrm{T}(0) = \mathrm{T}(0+0) = \mathrm{T}(0) + \mathrm{T}(0)$, so (by cancellation) $\mathrm{T}(0) = 0$. Now let $x \in \mathrm{V}$ and $c \in \mathbb{F}$. Then

$$\mathrm{T}(cx) = \mathrm{T}(cx+0) = c\mathrm{T}(x) + \mathrm{T}(0) = c\mathrm{T}(x),$$

   where we have again used (2) for the second equality, and the fact that $\mathrm{T}(0) = 0$ for the third.

3. Suppose T is linear. Then $\mathrm{T}(x-y) = \mathrm{T}(x+(-1)y) = \mathrm{T}(x)+(-1)\mathrm{T}(y) = \mathrm{T}(x)-\mathrm{T}(y)$, where condition (2) has been used for the second equality.

4. For $n \geq 1$ an integer, let us denote by $\mathcal{L}_n$ (linearity-n) the property

$$\mathcal{L}_n \iff \forall x_i \in \mathrm{V}\ \forall a_i \in \mathbb{F}\ \left[\mathrm{T}\left(\sum_{i=1}^{i=n} a_i x_i\right) = \sum_{i=1}^{i=n} a_i \mathrm{T}(x_i)\right].$$

   Translating a little, $\mathcal{L}_2$ says

$$\forall x_1, x_2 \in \mathrm{V}\ \forall a_1, a_2 \in \mathbb{F}\ \ [\mathrm{T}(a_1x_1 + a_2x_2) = a_1\mathrm{T}(x_1) + a_2\mathrm{T}(x_2)].$$

   First, setting $a_2 = 1$, we get that $\mathcal{L}_2$ implies condition (2), and therefore implies T is linear.
   So suppose T is linear. We want to prove that $\mathcal{L}_n$ is true for every n. Given some particular n, it's easy to prove - you just keep applying T's linearity to separate all the terms in the sum and pull all the coefficients through. But since we have to prove it for infinitely many cases, we'll use induction.
   Firstly notice that $\mathcal{L}_1$ is just the statement that T preserves multiplication, which is true by linearity.

Now assume $\mathcal{L}_m$ for some positive m. We want to prove $\mathcal{L}_{m+1}$. Let $x_i \in \mathrm{V}$ and $a_i \in \mathbb{F}$ for $1 \leq i \leq m+1$. Then

$$\mathrm{T}\left(\sum_{i=1}^{i=m+1} a_i x_i\right) = \mathrm{T}\left(\sum_{i=1}^{i=m} a_i x_i + a_{m+1}x_{m+1}\right) =$$

$$= \mathrm{T}\left(\sum_{i=1}^{i=m} a_i x_i\right) + \mathrm{T}(a_{m+1}x_{m+1}) = \mathrm{T}\left(\sum_{i=1}^{i=m} a_i x_i\right) + a_{m+1}\mathrm{T}(x_{m+1}).$$

Here we have used the linearity of T for the last two equalities. Now by inductive hypothesis, $\mathcal{L}_m$, so the $\mathrm{T}(\sum ax)$ term is equal to a $\sum a\mathrm{T}(x)$ term, i.e. the above is

$$= \sum_{i=1}^{i=m} a_i \mathrm{T}(x_i) + a_{m+1}\mathrm{T}(x_{m+1}) = \sum_{i=1}^{i=m+1} a_i \mathrm{T}(x_i).$$

Thus we have shown $\mathcal{L}_{m+1}$ is true. So by induction, $\forall n \mathcal{L}_n$.

Problem 13. Suppose $c_1v_1 + \ldots + c_kv_k = 0$. Then

$$0 = \mathrm{T}(0) = \mathrm{T}(\sum c_i v_i) = \sum c_i \mathrm{T}(v_i) = \sum c_i w_i.$$

(Here we have used property 4 from problem 7.) The w_i's form an independent set, so (as the above equation begins with 0), $c_i = 0$ for each i, and therefore the v_i's form an independent set.

Problem 14.

(a) Suppose T is 1-1 and S is an independent subset of V. We need to show $\mathrm{T}(S)$ is independent.
Let $w_i \in \mathrm{T}(S)$, $1 \leq i \leq n$, where the w_i's are distinct, and suppose $\sum c_i w_i = 0$ for some $c_i \in \mathbb{F}$.
Let v_i be such that $\mathrm{T}(v_i) = w_i$ (the v_i's exist by definition of $\mathrm{T}(S)$). Notice that the v_i's are distinct: if not, we have some $k < j$ such that $v_k = v_j$. But then $\mathrm{T}(v_k) = \mathrm{T}(v_j)$ so $w_k = w_j$. But the w_i's were chosen distinct, which means $k = j$, a contradiction. Now using property 4 of problem 7,

$$\mathrm{T}(\sum c_i v_i) = \sum c_i \mathrm{T}(v_i) = \sum c_i w_i = 0.$$

But T is 1-1, so $\mathbf{N}(\mathrm{T}) = \{0\}$, so we must have $\sum c_i v_i = 0$. As the v_i's are distinct elements of S, an independent set, we get $c_i = 0$ (for each i). Thus we have shown that $\mathrm{T}(S)$ is an independent set.
Now suppose T is not 1-1. Then $\mathbf{N}(\mathrm{T}) \supsetneq \{0\}$. Let $v \in \mathbf{N}(\mathrm{T})$, $v \neq 0$. Then $\{v\}$ is an independent set, but $\mathrm{T}(\{v\}) = \{0\}$, which is a dependent set. Therefore it is not true that T carries all independent sets onto independent sets.

(b) Suppose T is 1-1. If $S \subseteq \mathrm{V}$ is independent, then by (a), $\mathrm{T}(S)$ is independent. So suppose S is dependent. Let $v_1, \ldots, v_k \in S$ be distinct, mutually dependent vectors, and $c_1, \ldots, c_k \in \mathbb{F}$ be such that $c_1v_1 + \ldots + c_kv_k = 0$ non-trivially. Let $w_i = \mathrm{T}(v_i)$. Then the w_i's are distinct as T is 1-1 (if $w_k = w_j$ then $\mathrm{T}(v_k) = \mathrm{T}(v_j)$ but T is 1-1, so $v_k = v_j$, but the v_i's are distinct, so $k = j$). Applying T to the linear combination,

$$0 = \mathrm{T}(0) = \mathrm{T}(c_1v_1 + \ldots + c_kv_k) = c_1w_1 + \ldots + c_kw_k.$$

The c_i's were chosen to be non-trivial (not all 0), so they provide a non-trivial solution for the w_i's. $w_i \in \mathrm{T}(S)$, and as noted above, they are distinct, so $\mathrm{T}(S)$ is a dependent set.
Note that it really is important to show the distinctness of the vectors above. Consider the example of $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$, T projecting onto the x-axis. T is not 1-1, and there are dependent sets S such that $\mathrm{T}(S)$ is independent. Take S to be the line parallel to the y=axis, through $(1, 0)$. S is dependent as it has more than 2 elements. But $\mathrm{T}(S) = \{(1, 0)\}$, an independent set. Ignoring the issue of distinctness, the above proof (that S is dependent implies $\mathrm{T}(S)$ is dependent) goes through. So make sure you don't ignore it.

(c) As T is 1-1 and β is independent, by part (b), $\mathrm{T}(\beta)$ is independent. We need $\mathrm{span}(\beta) = \mathrm{W}$. But T is onto, so

$$\mathrm{W} = \mathrm{T}(\mathrm{V}) = \mathrm{T}(\mathrm{span}(\beta)) = \mathrm{T}\left(\{\sum c_i v_i | c_i \in \mathbb{F}\}\right) =$$

$$= \{\mathrm{T}(\sum c_i v_i) | c_i \in \mathbb{F}\} = \{\sum c_i \mathrm{T}(v_i) | c_i \in \mathbb{F}\} = \mathrm{span}(\mathrm{T}(\beta)).$$

Again we've used property 4 from problem 7.

Problem 15. Let $\mathrm{V} = \mathrm{P}(\mathbb{R})$. First I'll find a representation of T that it easier to deal with. Let $f \in \mathrm{V}$. We can express f in terms of the standard basis (as in problem 5), i.e.

$$f = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$. Note that the x^i's are basis elements. Now

$$\mathrm{T}(f)(r) = \int_0^r f(t)dt = \int_{t=0}^{t=r} (a_n t^n + \ldots + a_1 t + a_0)dt =$$

$$= \frac{a_n}{n+1} t^{n+1} + \ldots + a_0 t + c|_{t=0}^{t=r} = \frac{a_n}{n+1} r^{n+1} + \ldots + a_0 r.$$

Thus we have

$$\mathrm{T}(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) = \frac{a_n}{n+1} x^{n+1} + \ldots + a_0 x. \tag{3}$$

From now on we can use this as our definition of T. Using this, linearity of T is then easy to show.

Let us consider 1-1ness. It suffices to prove that $\mathbf{N}(\mathrm{T}) = \{0\}$, by Theorem 2.4. Suppose $\mathrm{T}(f) = 0$, where f is as above. So the polynomial on the right side of (3) is the 0 function. Hence the coefficients $\frac{a_i}{i+1} = 0$ (as the x^i's are independent, or in calculus language, the only polynomial that is constantly 0 is the polynomial with all coefficients 0), and so $a_i = 0$ for all i. Therefore $f = 0$. So we have shown $\mathbf{N}(\mathrm{T})$ is trivial, as required.

Now to see T is not onto. Inspecting (3), we can see that the constant term (the coefficient of $\mathbf{1}$) is 0. As this is true for all elements of $\mathbf{Rg}(\mathrm{T})$, we know $\mathbf{1} \notin \mathbf{Rg}(\mathrm{T})$, so T is not onto.

Problem 16. Good-o, here we can assume T is linear. It's easy to see T is not 1-1, as the derivative of any constant function is 0. So we just need ontoness. Here we can use that correspondence between integral and derivative. Let T' be the transformation from problem 15 (the integral). Then for any $f \in \mathrm{P}(\mathbb{R})$, $\mathrm{T}(\mathrm{T}'(f)) = f$. This is easily seen using the form of $\mathrm{T}'(f)$ in (3), and differentiating that function. All the $\frac{a_i}{i+1}$ terms return to a_i's when we "bring down the power", and we get f back. Thus $f \in \mathbf{Rg}(\mathrm{T})$, and as f was arbitrary, $\mathbf{Rg}(\mathrm{T}) = \mathrm{P}(\mathbb{R})$.

Problem 17.

(a) We have

$$\mathrm{rank}(\mathrm{T}) \leq \mathrm{rank}(\mathrm{T}) + \mathrm{nullity}(\mathrm{T}) = \dim(\mathrm{V}) < \dim(\mathrm{W}),$$

where the first inequality is because dimension is non-negative, the equality is the Dimension Theorem, and the last inequality is by hypothesis. Putting it together, $\mathrm{rank}(\mathrm{T}) < \dim(\mathrm{W})$, so by the fact following problem 1, T is not onto.

(b) If T were 1-1, $\mathrm{nullity}(\mathrm{T}) = 0$ (by the fact again), so $\dim(\mathrm{V}) = \mathrm{rank}(\mathrm{T})$ by the Dimension Theorem. But $\mathrm{rank}(\mathrm{T}) \leq \dim(\mathrm{W})$ as $\mathbf{Rg}(\mathrm{T})$ is a subspace of W, which contradicts $\dim(\mathrm{V}) > \dim(\mathrm{W})$.

Problem 18. To invent an example satisfying some particular property, one can often be motivated by experimenting with transformations in $\mathbb{R}^n$, particularly in $\mathbb{R}^2$ and $\mathbb{R}^3$. This problem actually requires us to have $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ anyway. Linear transformations in $\mathbb{R}^2$ can be constructed from projections onto lines, reflections about lines, rotations and scaling. Any combination of these can be made (one following the other), and the resulting transformation will still be linear (how does one prove that?) In this case, also notice that $\mathbf{N}(\mathrm{T}) = \mathbf{Rg}(\mathrm{T}) \implies \mathrm{nullity}(\mathrm{T}) = \mathrm{rank}(\mathrm{T})$, and combining this with the Dimension Theorem and $\mathrm{V} = \mathbb{R}^2$, we're forced to have $\mathrm{nullity}(\mathrm{T}) = \mathrm{rank}(\mathrm{T}) = 1$. The 1-dimensional subspaces of $\mathbb{R}^2$ are lines through the origin. So we need some line which is both the nullspace and range of T. As all such lines are just a rotation away from one another, it won't matter what line we work with, so let's aim for the x-axis.

Projection onto the y-axis has nullspace the x-axis. But its range is the y-axis, so it doesn't work. But as I mentioned, we can compose it with another map. There are a maps which swap the x-axis and y-axis, such as reflection about the line $x = y$, or rotation by 90°. If we first project onto the y-axis, and then rotate (say clockwise, 90°), the resulting range will be the x-axis. Also, rotation is 1-1, so it won't change the nullspace. So we have an example.

Let $\mathrm{Proj}_y : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the y-axis (so $\mathrm{Proj}_y(a, b) = (0, b)$) and $\mathrm{Rot} : \mathbb{R}^2 \to \mathbb{R}^2$ be clockwise rotation by 90° (so $\mathrm{Rot}(a, b) = (b, -a)$). Let $\mathrm{T} = \mathrm{Rot} \circ \mathrm{Proj}_y$ (this notation means $\mathrm{T}(a, b) = \mathrm{Rot}(\mathrm{Proj}_y(a, b))$). So $\mathrm{T}(a, b) = (b, 0)$. Then $\mathbf{N}(\mathrm{T}) = \mathbf{Rg}(\mathrm{T})$, the x-axis.

Problem 19. Considering the previous example, we also could have rotated counter-clockwise. This would have resulted in $\mathrm{U}(a, b) = (-b, 0)$. The T above and this U suffice. Notice that we just have $\mathrm{U} = -\mathrm{T}$ (i.e. $\mathrm{U}(x) = -\mathrm{T}(x)$ for all x). We can get more examples by taking any linear transformation T, letting $a \neq 0$, $a \neq 1$ (this needs $\mathrm{char}(\mathbb{F}) \neq 2$), and letting $\mathrm{U} = a\mathrm{T}$ (defined similarly to $-\mathrm{T}$).

There are ways where T and U seem less similar though. For instance, say $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^3$, and $\mathbf{Rg}(\mathrm{T})$ is a plane P. Suppose $\mathrm{U}' : P \to P$ is linear, and is 1-1 and onto (say reflection about some line in the plane, or rotation about the origin). Then letting $\mathrm{U} = \mathrm{U}' \circ \mathrm{T}$, we have a more interesting example.

Problem 20. First we show $\mathrm{T}(\mathrm{V}_1)$ is a subspace of W. As $\mathrm{T} : \mathrm{V} \to \mathrm{W}$, $\mathrm{T}(\mathrm{V}_1) \subseteq \mathrm{W}$. As $0 \in \mathrm{V}_1$ and $\mathrm{T}(0) = 0$ by linearity, we have $0 \in \mathrm{T}(\mathrm{V}_1)$.

Let $w_1, w_2 \in \mathrm{T}(\mathrm{V}_1)$ and $c \in \mathbb{F}$. We need to show $cw_1 + w_2 \in \mathrm{T}(V_1)$. Let $v_1, v_2 \in \mathrm{V}_1$ be such that $\mathrm{T}(v_1) = w_1$ and $\mathrm{T}(v_2) = w_2$ (these exist by definition of $\mathrm{T}(\mathrm{V}_1)$. By linearity, we have

$$T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2. \quad (4)$$

But V_1 is a subspace, so $cv_1 + v_2 \in V_1$, so $T(cv_1 + v_2) \in T(V_1)$. So by (4), $cw_1 + w_2 \in T(V_1)$, as desired.

Now we show $T^{-1}(W_1) = \{v \in V | T(v) \in W_1\}$ is a subspace of V. Firstly, $T(0) = 0 \in W_1$, so $0 \in T^{-1}(W_1)$.

Now let $x, y \in T^{-1}(W_1)$ and $c \in \mathbb{F}$. We need $cx + y \in T^{-1}(W_1)$, which is equivalent to $T(cx + y) \in W_1$. But $T(cx + y) = cT(x) + T(y)$ and we know $T(x), T(y) \in W_1$ because $x, y \in T^{-1}(W_1)$. As W_1 is a subspace, we have $cT(x) + T(y) \in W_1$, so $T(cx + y) \in W_1$, as required.

Problem 21.

(a) Let $y, z \in V$ and $b \in \mathbb{F}$. $T(y + bz) =$

$$T(y_1 + bz_1, y_2 + bz_2, \ldots) = (y_2 + bz_2, y_3 + bz_3, \ldots) = (y_2, y_3, \ldots) + b(z_2, z_3, \ldots)$$

Showing U is linear is almost the same.

(b) Clearly T is not 1-1 as two sequences y, z may differ only on their first entry. T is onto because given $y \in V$, right-shifting then left-shifting the result gives y back, i.e. $T(U(y)) = y$. Thus $y \in \mathbf{Rg}(T)$.

(c) U is not onto because it doesn't produce the sequence $(1, 0, 0, \ldots)$. It is 1-1 because if $U(y) = U(z)$ then $T(U(y)) = T(U(z))$, and as mentioned above, $T(U(y)) = y$ and likewise for z, so $y = z$.

Problem 22. Given $v = (x, y, z) \in \mathbb{R}^3$, we have $v = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$. Therefore

$$T(v) = T(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1),$$

using linearity for the second equality. So we have to set $a = T(1, 0, 0), b = T(0, 1, 0), c = T(0, 0, 1)$, and then we get $T(x, y, z) = ax + by + cz$ as required.

Notice we had no choice about what a, b, c were - they were chosen by T. Also notice we can also write the action of T as matrix multiplication:

$$T(x, y, z) = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

For the linear $T : \mathbb{F}^n \to \mathbb{F}$ case, we do the same thing, and get some $a_i \in \mathbb{F}$, such that $T(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n$.

Now suppose $T : \mathbb{F}^n \to \mathbb{F}^m$. Let $\{e^{(1)}, \ldots, e^{(n)}\}$ be the standard basis for $\mathbb{F}^n$ (so $e^{(2)} = (0, 1, 0, 0, \ldots, 0)$, etc). We can do exactly the same thing as the $\mathbb{R}^3 \to \mathbb{R}$ case, and express any input vector as a linear combination of the $e^{(i)}$'s. So we'll need to know the value T takes on these vectors. $T(e^{(i)}) \in \mathbb{F}^m$ for each i, so it is some m-tuple. Let's denote it $a^{(i)}$, so $a^{(i)} = T(e^{(i)})$. (This could be a little deceptive though - remember the $a^{(i)}$'s are not scalars in this case, but m-tuples of scalars. Now by linearity, we get

$$T(x_1, \ldots, x_n) = T(x_1e^{(1)} + \ldots + x_ne^{(n)}) = x_1a^{(1)} + \ldots + x_na^{(n)}.$$

So we've found a generalization of the above - instead of scalars, we just have vectors. Again, T forced the choice of the $a^{(i)}$'s on us. But let's try writing out all the vectors explicitly. Say

$a^{(i)} = (a_{1i}, \dots, a_{mi})$ (remember they were m-tuples). Then, writing the vectors as columns,

$$\mathrm{T}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where at the bottom we have the multiplication of a matrix and vector. So any linear transformation $\mathrm{T} : \mathbb{F}^n \to \mathbb{F}^m$ can be represented in the form $\mathrm{T}(x) = Ax$, where A is the matrix determined as above. As the choice of the $a^{(i)}$'s was determined by T, and the matrix is composed of the components of the $a^{(i)}$'s, the matrix is completely determined by T.

Problem 25. To do this problem, make sure you read the definition on the previous page, before problem 24. When a vector space $\mathrm{V} = \mathrm{W}_1 \oplus \mathrm{W}_2$, every vector in V has a unique representation as $v = w_1 + w_2$, where $w_i \in \mathrm{W}_i$. Then the projection onto W_1 along W_2 just sends v to w_1. You might think of the vectors in V being represented with two components (though those components are vectors, not scalars), and we're projecting onto one of the components. This definition is abstract, though, and you can also think about it as follows. Consider the $xy-$plane in $\mathbb{R}^3$ and a line L through the origin, not lying in the $xy-$plane. We can define projection Pr onto the $xy-$plane along L by sending any point p to the $xy-$plane along the line through p, parallel to L. This line intersects the $xy-$plane at some unique point, and we define $\mathrm{Pr}(p)$ to be that point of intersection. This is a case of the general definition.

(a) First we need to check that this makes sense, i.e. that $\mathrm{V} = \mathbb{R}^3 = xy-$plane $\oplus\, z-$axis. But by definition, this is just that $\mathbb{R}^3 = xy-$plane $+\, z-$axis and $xy-$plane $\cap\, z-$axis $= \{0\}$. The first of these comes from noting that $(x, y, z) = (x, y, 0) + (0, 0, z) \in xy-$plane $+\, z-$axis. The second is easy too.

Now, given $v = (x, y, z) \in \mathbb{R}^3$, $v = (x, y, 0) + (0, 0, z)$, and $(x, y, 0) \in xy-$plane and $(0, 0, z) \in z-$axis. By definition, $\mathrm{T}(x, y, z) = (x, y, 0)$, so T is the projection on the $xy-$plane along the $z-$axis.

(b) We already know $\mathbb{R}^3 = z-$axis $\oplus\, xy-$plane from part (a) (by commutativity of $+$ and $\cap$, the definition is symmetric, so $\mathrm{V} = \mathrm{W}_1 \oplus \mathrm{W}_2 \iff \mathrm{V} = \mathrm{W}_2 \oplus \mathrm{W}_1$). Given $v = (x, y, z) \in \mathbb{R}^3$, we have $v = (0, 0, z) + (x, y, 0)$ is the $z-$axis $+\, xy-$plane representation of v, so we need $\mathrm{T}(x, y, z) = (0, 0, z)$.

(c) As the line L does not lie in the $xy-$plane, it's easy to show $xy-$plane $\oplus\, L = \mathbb{R}^3$. Given $(a, b, c) \in \mathbb{R}^3$, $\mathrm{T}(a, b, c) = (a - c, b, 0)$, which is certainly in the $xy-$plane, so that's fine. Now consider $(a, b, c) - \mathrm{T}(a, b, c) = (a, b, c) - (a - c, b, 0) = (c, 0, c)$. This is in L. So $(a, b, c) = v + w$ where $v = \mathrm{T}(a, b, c) \in xy-$plane and $w = (c, 0, c) \in L$, as required.

Problem 35.

(a) We just need to verify $\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T}) = \{0\}$. By exercise 29 of 1.6,

$$\dim(\mathrm{V}) = \dim(\mathbf{Rg}(\mathrm{T})) + \dim(\mathbf{N}(\mathrm{T})) - \dim(\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T}))$$
$$= \mathrm{rank}(\mathrm{T}) + \mathrm{nullity}(\mathrm{T}) - \dim(\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T})).$$

So by the Dimension Theorem, $\dim(\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T})) = 0$, so $\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T}) = \{0\}$. Finite-dimensionality is required by both the Dimension Theorem and exercise 29.

(b) By the same result,

$$\dim(\mathbf{Rg}(\mathrm{T}) + \mathbf{N}(\mathrm{T})) = \dim(\mathbf{Rg}(\mathrm{T})) + \dim(\mathbf{N}(\mathrm{T})) - \dim(\mathbf{Rg}(\mathrm{T}) \cap \mathbf{N}(\mathrm{T})),$$

but the last term is 0 by hypothesis. Combining this with the Dimension Theorem,

$$\dim(\mathbf{Rg}(\mathrm{T}) + \mathbf{N}(\mathrm{T})) = \dim(\mathrm{V}).$$

But $\mathbf{Rg}(\mathrm{T}) + \mathbf{N}(\mathrm{T})$ is a subspace of V, so by Theorem 1.11, $\mathrm{V} = \mathbf{Rg}(\mathrm{T}) + \mathbf{N}(\mathrm{T})$, which is all we needed. Finite-dimensionality was used in the same way here.

Problem 38. Let $r + si, x + yi \in \mathbb{C}$. Then

$$\mathrm{Conj}((r + si) + (x + yi)) = \mathrm{Conj}((r + x) + (s + y)i) = (r + x) - (s + y)i =$$
$$= (r - si) + (x - yi) = \mathrm{Conj}(r + si) + \mathrm{Conj}(x + yi).$$

However, $i\,\mathrm{Conj}(1) = i.1 = i$, but $\mathrm{Conj}(i.1) = \mathrm{Conj}(i) = -i$. So Conj does not preserve multiplication by complex scalars.