MATH 110: LINEAR ALGEBRA
HOMEWORK #4

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§2.2: The Matrix Representation of a Linear Transformation

Problem 1.

(a) True. This is a consequence of $\mathcal{L}(V, W)$ being a vector space.

(b) True. Suppose $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{w_1, \cdots, w_m\}$. Let $A = [T]_\beta^\gamma = [U]_\beta^\gamma$. Then for each v_j,

$$T(v_j) = \sum_{i=1}^{m} A_{i,j} w_i = U(v_j).$$

Since T and U agree on β they must be equal.

(c) False. By definition it will be a $n \times m$ matrix.

(d) True. This is Theorem 2.8(a)

(e) True. This is proved in Theorem 2.7.

(f) False. If $V \neq W$ then we can't have equality (since the functions in both sets have different domains, and hence are different).

Problem 3. We have a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. We have bases $\beta = \{(1, 0), (0, 1)\}$ and $\alpha = \{(1, 2), (2, 3)\}$ of $\mathbb{R}^2$, and basis $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$ of $\mathbb{R}^3$.

After computing $T(1, 0)$ and $T(0, 1)$ in terms of the basis γ we find

$$\begin{aligned} T(1,0) &= (1,1,2) = \frac{-1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3) \\ T(0,1) &= (-1,0,1) = -(1,1,0) + (0,1,1) + 0(2,2,3). \end{aligned}$$

Therefore

$$[T]_\beta^\gamma = \begin{pmatrix} \frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

We now compute $T(1, 2)$ and $T(2, 3)$ in terms of the basis γ (using our earlier calculations).

$$\begin{aligned} T(1,2) &= T(1,0) + 2T(0,1) \\ &= \left(\frac{-1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)\right) + 2\left(-(1,1,0) + (0,1,1) + 0(2,2,3)\right) \\ &= \frac{-7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3) \end{aligned}$$

Date: September 29th.

$$\begin{aligned}
T(2,3) &= 2T(1,0)+3T(0,1)\\
&= 2\left(\frac{-1}{3}(1,1,0)+0(0,1,1)+\frac{2}{3}(2,2,3)\right)+3\left(-(1,1,0)+(0,1,1)+0(2,2,3)\right)\\
&= \frac{-11}{3}(1,1,0)+3(0,1,1)+\frac{4}{3}(2,2,3)
\end{aligned}$$

Therefore

$$[T]_\alpha^\gamma = \begin{pmatrix} \frac{-7}{3} & \frac{-11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

Problem 4. Evaluating the basis elements of $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ by T we get:

$$\begin{aligned}
T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 1\\
T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= 1+x^2\\
T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 0\\
T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= 2x
\end{aligned}$$

Therefore we have

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $\gamma = \{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$.

Problem 6. Given vector spaces V and W over a field F, let $\mathcal{L}(V,W)$ be the set of linear transformations from V into W. We define addition and scalar mulitplication in $\mathcal{L}(V,W)$ as described on page 82. We must proof Theorem 2.7(b), which says that under these operations $\mathcal{L}(V,W)$ is itself a vector space.

The set $\mathcal{L}(V,W)$ contains the zero transformation T_0. In Theorem 2.7(a) it is show that $\mathcal{L}(V,W)$ is closed under addition and scalar multiplication. What is left to check is the vector space axioms. All of the axioms are easy to check, and they come down to the equivalent axiom of the vector space V which we will implicitly use.

Take arbitrary $S, T, U \in \mathcal{L}(V,W)$ and scalars a, b. To check that the following functions are equal it suffices to show that they agree on an arbitrary element $v \in V$. (see page 7 for the axioms)

(VS 1) :

$$(S+T)(v) = S(v)+T(v) = T(v)+S(v) = (T+S)(v)$$

(VS 2) :

$$\begin{aligned} ((S+T)+U)(v) &= (S+T)(v)+U(v) \\ &= (S(v)+T(v))+U(v) \\ &= S(v)+(T(v)+U(v)) \\ &= S(v)+(T+U)(v) \\ &= (S+(T+U))(v) \end{aligned}$$

(VS 3) :

$$(T+T_0)(v) = T(v)+T_0(v) = T(v)+0 = T(v)$$

(VS 4) : The map $B := -T$ is in $\mathcal{L}(V,W)$, since we know $\mathcal{L}(V,W)$ is closed under scalar multiplication.

$$(T+B)(v) = T(v)+B(v) = T(v)-T(v) = 0 = T_0(v)$$

(VS 5) :

$$(1 \cdot T)(v) = 1 \cdot T(v) = T(v)$$

(VS 6) :

$$((ab)T)(v) = (ab)T(v) = a(bT(v)) = a(bT)(v) = (a(bT))(v)$$

(VS 7) :

$$(a(S+T))(v) = a(S+T)(v) = a(S(v)+T(v)) = aS(v)+aT(v) = (aS)(v)+(aT)(v) = (aS+aT)(v)$$

(VS 8) :

$$((a+b)T)(v) = (a+b)T(v) = aT(v)+bT(v) = (aT)(v)+(bT)(v) = (aT+bT)(v)$$

Problem 9. In this exercise $V = \mathbb{C}$, a *real* vector space. Define $T : V \to V$ by $T(z) = \overline{z}$.

Given any complex numbers $z_1 = x_1+iy_1, z_2 = x_2+iy_2 \in V$ (where $x_1, x_2, y_1, y_2 \in \mathbb{R}$) and scalar $c \in \mathbb{R}$, we have

$$T(cz_1+z_2) = T((cx_1+x_2)+i(cy_1+y_2)) = (cx_1+x_2)-i(cy_1+y_2) = c(x_1-iy_1)+(x_2-iy_2) = cT(z_1)+T(z_2).$$

Since z_1, z_2 and c were arbitrary, we have that T is a linear map (of real vector spaces).

Since $T(1) = 1$ and $T(i) = -i$, we have $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $\beta = \{1, i\}$.

Problem 10. We have a basis $\beta = \{v_1, \cdots, v_n\}$ of V, and a linear map $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \cdots, n$ (where we have $v_0 = 0$ by convention). The $n \times n$ matrix $A = [T]_\beta$ is defined to be the unique matrix such that

$$T(v_j) = \sum_{i=1}^{n} A_{i,j} v_i \text{ for } 1 \leq j \leq n.$$

Hence for $1 \leq i, j \leq n$,

$$([T]_\beta)_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ or } i = j-1 \ , \\ 0 & \text{otherwise.} \end{cases}$$

This is a matrix with 1's on the main diagonal, 1's on the diagonal above that, and 0's everywhere else: $\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix}$

Problem 11. Choose a basis $\{v_1, \cdots, v_k\}$ for the vector space W. We can extend this to a basis $\beta = \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ of V.

When $1 \leq j \leq k$, by construction $v_j \in W$ and hence $T(v_j) \in W$ (this is where we use that W is T-invariant). Since $T(v_j) \in W$, we can write

$$T(v_j) = \sum_{i=1}^{k} M_{i,j} v_i, \tag{1}$$

for some scalars $M_{i,j}$.

However we know that the $([T]_\beta)_{i,j}$ are the *unique* scalars such that

$$T(v_j) = \sum_{i=1}^{n} ([T]_\beta)_{i,j} v_i.$$

So by comparison wtih (1) we see that

$$([T]_\beta)_{i,j} = 0,$$

when $1 \leq j \leq k$ and $k+1 \leq i \leq n$.

This is the same as saying that $[T]_\beta$ is of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

Problem 14. Suppose that there we have a linear combination

$$a_1 T_1 + \cdots a_n T_n = 0,$$

with $a_1, \cdots, a_n \in \mathbb{R}$.

Fix a $j \in \{1, \cdots, n\}$. Define the polynomial $g(x) = (a_1 T_1 + \cdots a_n T_n)(x^j)$, which has constant term $j! a_j$. Thus $g(0) = j! a_j$. However, by assumption $a_1 T_1 + \cdots a_n T_n = 0$ so $g(x) = 0$. This implies that $j! a_j = 0$, which shows that $a_j = 0$.

Since j was arbitrary we have shown that $a_1 = \cdots = a_n = 0$. Therefore $T_1, \cdots, T_n$ are linearly independent.

Problem 16. Let $n = \dim V = \dim W$. Let $T : V \to W$ be a linear map.

Choose a basis $u_1, \cdots, u_k$ of $N(T)$. We can then extend this to some basis

$$\beta = \{v_1, \cdots, v_{n-k}, u_1, \cdots, u_k\}$$

of V.

Claim: $T(v_1), \cdots, T(v_{n-k})$ are linearly independent

Proof. Suppose we have a linear combination

$$a_1 T(v_1) + \cdots + a_{n-k} T(v_{n-k}) = 0.$$

This implies that $T(a_1 v_1 + \cdots + a_{n-k} v_{n-k}) = 0$, since T is linear. Thus $a_1 v_1 + \cdots + a_{n-k} v_{n-k} \in N(T)$, so we can write it as

$$a_1 v_1 + \cdots + a_{n-k} v_{n-k} = b_1 u_1 + \cdots + b_k u_k \tag{2}$$

for some scalars b_j. Since β is a basis of V (and in particular linearly independent), we find that all of coefficients of (2) are zero. Therefore $a_1 = \cdots = a_{n-k} = 0$, which show the desired independence. $\square$

Define $w_1 = T(v_1), \cdots, w_{n-k} = T(v_{n-k})$. These vectors are linearly independent by our claim, and we can extend to a basis of $\gamma = \{w_1, \cdots, w_{n-k}, w_{n-k+1}, \cdots, w_n\}$.

By construction, $T(v_i) = w_i$ and $T(u_i) = 0$. Therefore we have

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix},$$

where the first $n-k$ diagonal elements are 1's and all the other entries are zeros. In particular, the matrix $[T]_\beta^\gamma$ is diagonal.

We now prove the dimension theorem in the case $n = \dim V = \dim W$.
Let $T : V \to W$ be a linear map. From above we know that we can choose a bases β of V and γ of W such that the matrix $A := [T]_\beta^\gamma$ is diagonal as above. After choosing these bases we see that the map $T : V \to W$ can be viewed as the linear map $L_A : F^n \to F^n$ (this is the content of the next section). In particularly, it is apparent that

$$\text{nullity}(T) = \text{nullity}(L_A) = \text{nullity}(A) \text{ and } \text{rank}(T) = \text{rank}(L_A) = \text{rank}(A)$$

. Thus it suffices to prove that for an $n \times n$ *diagonal* matrix A,

$$\text{nullity}(A) + \text{rank}(A) = n.$$

For a diagonal matrix A, let r be the number of zeros on the main diagonal of A. It is then easy to see that $\text{nullity}(A) = r$ and $\text{rank}(A) = n - r$. Thus $\text{nullity}(A) + \text{rank}(A) = k + (n-k) = n$.

§2.3: Composition of linear transformations and matrix multiplication

Problem 1.

(a) False. This is an incorrect statement of Theorem 2.11. If $\dim W \neq \dim Z$ then the expression $[T]_\alpha^\beta [U]_\beta^\gamma$ doesn't make sense [you cannot multiply matrices unless their dimensions match up]. Note that even if $\dim W = \dim Z$ then the statement is still false, since in general matrices don't commute.
(b) True. This is Theorem 2.14.
(c) False. We have bases α and β of V and W respectively. Since U is a map from W to Z, the symbol $[U]_\alpha^\beta$ will not always make sense (since α (resp. β) need not be a basis of W (resp. Z)).
(d) True. If $\alpha = \{v_1, \cdots, v_n\}$ then, $I_v(v_i) = v_i$. This shows that $[I_V]_\alpha = I$.
(e) False. If $V \neq W$ then T^2 is not defined. However, even if $V = W$ the statement may fail.
Consider the easiest counter example: define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x$. We have bases $\alpha = \{2\}$ and $\beta = \{1\}$ fo $\mathbb{R}$. Then $[T^2]_\alpha^\beta = [T]_\alpha^\beta = [2]$, but $([T]_\alpha^\beta)^2 = [2]^2 = [4]$.
(f) False. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $A^2 = I$
(g) False. If T is a map of the form $F^n \to F^m$ then Theorem 2.15 shows that $T = L_A$ for some matrix A. Otherwise, it is unclear what L_A means (for example, what does $L_A : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ mean?).
(h) False. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $A^2 = 0$
(i) True. This is Theorem 2.15(c). (assuming A and B have the same dimensions!)
(j) True. This is the definition of I (see page 89).

Problem 3. We have linear maps $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(f(x)) = f'(x)(3 + x) + 2f(x)$ and $U(a + bx + cx^2) = (a + b, c, a - b)$. We also have bases $\beta = \{1, x, x^2\}$ and $\gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

(a)

$$\begin{aligned} U(1) &= (1,0,1) = (1,0,0) + 0 \cdot (0,1,0) + (0,0,1) \\ U(x) &= (1,0,-1) = (1,0,0) + 0 \cdot (0,1,0) - (0,0,1) \\ U(x^2) &= (0,1,0) = 0 \cdot (1,0,0) + (0,1,0) + 0 \cdot (0,0,1) \end{aligned}$$

Thus

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$$\begin{aligned} T(1) &= 2 = 2 \cdot 1 + 0x + 0x^2 \\ T(x) &= 3 + 3x = 3 \cdot 1 + 3x + 0x^2 \\ T(x^2) &= 6x + 4x^2 = 0 \cdot 1 + 6x + 4x^2 \end{aligned}$$

Thus

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

$$\begin{aligned} UT(1) &= U(2) = (2,0,2) = 2(1,0,0) + 0(0,1,0) + 2(0,0,1) \\ UT(x) &= U(3+3x) = (6,0,0) = 6(1,0,0) + 0(0,1,0) + 0(0,0,1) \\ U(x^2) &= U(6x+4x^2) = (6,4,-6) = 6(1,0,0) + 4(0,1,0) - 6(0,0,1) \end{aligned}$$

Thus

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

We now verify Theorem 2.11, which says that $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$.

$$[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = [UT]_\beta^\gamma.$$

(a) Since $h = 3 - 2x + x^2$, we have $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. Since $U(h(x)) = (4,-2,2)$ we have $[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

We now verify Theorem 2.14, which says that $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta$.

$$[U]_\beta^\gamma [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = [U(h(x))]_\gamma.$$

Problem 8. We now prove Theorem 2.10 (see page 87 for the statements). To show that two functions are the same it suffices to show that they agree on their domains.

(a) Take any $v \in V$. Then

$$\begin{aligned} (T(U_1+U_2))(v) &= T((U_1+U_2)(v)) = T(U_1(v) + U_2(v)) = T(U_1(v)) + T(U_2(v)) \\ &= (TU_1)(v) + (TU_2)(v) = (TU_1 + TU_2)(v). \end{aligned}$$

Since this holds for all $v \in V$, we have $T(U_1+U_2) = TU_1 + TU_2$.

Take any $v \in V$. Then

$$((U_1+U_2)T)(v) = (U_1+U_2)(T(v)) = U_1(T(v)) + U_2(T(v)) = (U_1T)(v) + (U_2T)(v) = (U_1T + U_2T)(v).$$

Since this holds for all $v \in V$, we have $(U_1+U_2)T = U_1T + U_2T$.

(b) This part does not use linear algebra. It is a consequence of the general fact that the composition of functions is associative.

(c) For any $v \in V$,

$$(TI)(v) = T(I(v)) = T(v).$$

Since TI and T agree on their domains, they must be equal. Similarily we can prove that $IT = T$.

(d) Take any $v \in V$.

$$(a(U_1U_2))(v) = a((U_1U_2)(v)) = a(U_1(U_2(v)))$$
$$((aU_1)U_2)(v) = (aU_1)(U_2(v)) = a(U_1(U_2(v)))$$
$$(U_1(aU_2))(v) = U_1((aU_2)(v)) = U_1(aU_2(v)) = a(U_1(U_2(v)))$$

This shows that the functions $a(U_1U_2)$, $(aU_1)U_2$ and $U_1(aU_2)$ agree on all elements of V, therefore these functions are equal.

Theorem. (Generalization of Theorem 2.10) Let V, W, Z, X be vector spaces.

(a) Let $U_1, U_2 \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, Z)$,
$$T(U_1 + U_2) = TU_1 + TU_2.$$
Let $U_1, U_2 \in \mathcal{L}(W, Z)$ and $T \in \mathcal{L}(V, W)$,
$$(U_1 + U_2)T = U_1T + U_2T.$$
(b) For $U_2 \in \mathcal{L}(V, W)$, $U_1 \in \mathcal{L}(W, Z)$ and $T \in \mathcal{L}(Z, X)$,
$$T(U_1U_2) = (TU_1)U_2.$$
(c) For $T \in \mathcal{L}(V, W)$,
$$TI_V = I_WT = T.$$
(Where I_V and I_W are the identity maps on V and W respectively.)

(d) For $U_2 \in \mathcal{L}(V, W)$, $U_1 \in \mathcal{L}(W, Z)$ and a scalar a,
$$a(U_1U_2) = (aU_1)U_2 = U_1(aU_2).$$

The proofs given before carry over verbatim in this more general situation.

Problem 12. We are give linear maps $T : V \to W$ and $U : W \to Z$.

(a) Suppose that UT is one-to-one, we will now show that T is one to one.

Take any $v \in N(T)$. Then $(UT)(v) = U(T(v)) = U(0) = 0$, and since UT is one-to-one this implies that $v = 0$. Therefore we must have $N(T) = 0$, and hence T is one-to-one.

Note that U need not be one-to-one. Consider the following example: Define $T : \mathbb{R} \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}$ by $T(x) = (x, 0)$ and $U(x, y) = x$. The composition UT is the identity function on $\mathbb{R}$ and is in particular one-to-one. However, the function U is not one-to-one since $U(0, 1) = 0$.

(b) Suppose that UT is onto, we will now show that U is onto.

Take any $z \in Z$. Since UT is onto we know there is a $v \in V$ such that $(UT)(v) = z$. This shows that $z = U(T(v))$ is in the range of U. Since $z \in Z$ was arbitary we find that $R(U) = Z$; therefore U is onto.

Note that T need not be onto. Consider the example from part (a): Define $T : \mathbb{R} \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}$ by $T(x) = (x, 0)$ and $U(x, y) = x$. The composition UT is the identity function on $\mathbb{R}$ and is in particular onto. However, the function T is not onto since $(0, 1) \notin R(T)$.

(c) Suppose that U and T are one-to-one. We show that UT is one-to-one also.

Take any $v \in N(UT)$. Since $U(T(v)) = (UT)(v) = 0$, we find that $T(v) = 0$ [here we used that U is one-to-one]. But this then implies that $v = 0$ since T is also one-to-one. Therefore

$N(UT) = 0$, and hence UT is one-to-one.

Now suppose that U and T are onto. We show that UT is onto also.

Take any $z \in Z$. Since U is onto, there is a $w \in W$ such that $U(w) = z$. Since T is onto, there is a $v \in V$ such that $T(v) = w$. Combining everything we find that

$$(UT)(v) = U(T(v)) = U(w) = z.$$

Since $z \in Z$ was arbitrary we find that $R(UT) = Z$. Therefore UT is onto as desired.

Problem 13. Let A and B be $n \times n$ matrices.

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{i,i} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{i,k} B_{k,i} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{k,i} A_{i,k} \\
&= \sum_{k=1}^{n} (BA)_{k,k} \\
&= \operatorname{tr}(BA)
\end{aligned}$$

Recall that A^t is the $n \times n$ matrix such that $(A^t)_{i,j} = A_{j,i}$.

$$\operatorname{tr}(A^t) = \sum_{i=1}^{n} (A^t)_{i,i} = \sum_{i=1}^{n} A_{i,i} = \operatorname{tr}(A)$$