MATH 110: LINEAR ALGEBRA
HOMEWORK #12

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Problem 1.1. Since $Ty = \lambda y$, $y \in N(T - \lambda I)$. T is an upper-triangular matrix whose (i,i)-th entry is λ. So $T - \lambda I$ is upper-triangular; its (i,i)-th entry is zero, and all the other diagonal entries are non-zero (the last result follows from the fact that all eigenvalues of T are distinct). Write $T - \lambda I$ in the form:

$$\begin{pmatrix} T_1 & A & B \\ 0 & 0 & C \\ 0 & 0 & T_2 \end{pmatrix},$$

where T_1, A, B, C, T_2 are $(i-1)\times(i-1)$, $(i-1)\times 1$, $(i-1)\times(n-i)$, $1\times(n-i)$ and $(n-i)\times(n-i)$ matrices respectively. Also, T_1 and T_2 are invertible and upper-triangular.

Suppose $(y_1,\ldots,y_n)^t$ is in the null space of this matrix. Then we get $y_{i+1} = y_{i+2} = \cdots = y_n = 0$. To get a non-trivial eigenvector, let $y_i = 1$. This gives: $T_1(y_1,\ldots,y_{i-1})^t + A = 0$. Hence $(y_1,\ldots,y_{i-1})^t = -T_1^{-1}A$.

Problem 1.2. Since Q is unitary, we have $QQ^* = I$. Hence Q is invertible, with inverse Q^*. Thus we have

$$A = QTQ^* = QTQ^{-1}.$$

Since $Ty = \lambda y$, we get

$$A(Qy) = QTQ^{-1}(Qy) = QTy = Q(\lambda y) = \lambda(Qy),$$

and Qy is an eigenvector of A with eigenvalue λ. Furthermore, $Qy \neq 0$ since Q is unitary (and hence invertible) and $y \neq 0$.

Problem 2.1. The condition $||u|| = 1$ gives $1 = ||u||^2 = \langle u, u\rangle = u^*u$. Hence, if $P = I - 2uu^*$, then we get

$$\begin{aligned} PP^* &= (I - 2uu^*)(I - 2uu^*)^* = (I - 2uu^*)(I^* - 2(u^*)^*u^*) \\ &= (I - 2uu^*)(I - 2uu^*) = I - 4uu^* + 4(uu^*)(uu^*). \end{aligned}$$

Since $u^*u = 1$, we have $(uu^*)(uu^*) = u(u^*u)u^* = uu^*$, and the above identity simplifies to $PP^* = I$.

Problem 2.2. Write $q = (q_1,\ldots,q_n)^t$. We may assume $n \geq 2$, otherwise q itself gives the desired 1×1 matrix.

Date: December 9th.

We first suppose q_1 is real and positive. Since $||q|| = 1$, we have $0 \leq q_1 \leq 1$. We claim that there is a u of norm 1 such that $P = I - 2uu^*$ has q as its first column. Note that the first column of P equals $(1 - 2u_1\overline{u}_1, -2u_2\overline{u}_1, \dots, -2u_n\overline{u}_1)^t$. Now, we consider two cases.

Case 1 : $q_1 = 1$, in which case $q_2 = \dots = q_n = 0$. Let u be any vector of norm 1 with first component $u_1 = 0$, e.g. $u = (0, \dots, 0, 1)^t$. Such a vector exists since $n \geq 2$. Then the first column of P would be $(1, 0, \dots, 0)^t = q$.

Case 2 : $q_1 < 1$. Equating the first component of q gives

$$q_1 = 1 - 2|u_1|^2 \implies |u_1| = \sqrt{\frac{1 - q_1}{2}}.$$

Let $u_1 = \sqrt{\frac{1-q_1}{2}}$ which is real and positive. Also, let $u_i = \frac{q_i}{-2\overline{u}_1} = \frac{q_i}{-2u_1}$. We see that if $u = (u_1, u_2, \dots, u_n)^t$, then the first column of $P = I - 2uu^*$ is precisely q.

Finally, we have to consider the general case when q_1 is possibly not real and positive. Let $c \in \mathbb{C}$, $|c| = 1$, such that cq_1 is real and positive. We have just proven that there is a unitary matrix P whose first column equals cq. Then the first column of $c^{-1}P$ equals q, and furthermore

$$(c^{-1}P)(c^{-1}P)^* = (c^{-1}P)(\overline{c}^{-1}P^*) = |c^{-1}|^2PP^* = 1,$$

since $|c| = 1$ and P is unitary.

Problem 3.1. Write $A = SJS^{-1}$, where J is a block-Jordan matrix:

$$J = \begin{pmatrix} J_{\lambda_1} & 0 & \dots & 0 \\ 0 & J_{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\lambda_m} \end{pmatrix}.$$

Suppose each J_{λ_i} is an $r_i \times r_i$ matrix. Now $A^k = SJ^kS^{-1}$ and J^k consists of $r_i \times r_i$ block matrices of the form $J_{\lambda_i}^k$. Thus $\lim_{k\to\infty} A^k$ exists iff $\lim_{k\to\infty} J^k$ exists, which holds iff $\lim_{k\to\infty} J_{\lambda_i}^k$ exists *for each i*.

(1) *First we prove: if $J_{\lambda_i}^k$ is bounded, then $|\lambda_i| < 1$, or $\lambda_i = 1$ and $r_i = 1$.* Now, $J_{\lambda_i}^k$ is an upper-triangular matrix whose diagonal entries are λ_i^k. Also, $\lim_{k\to\infty} \lambda_i^k$ exists iff $|\lambda_i| < 1$ or $\lambda_i = 1$. In the case where $\lambda_i = 1$, we get:

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & * & \dots & * \\ 0 & 1 & k & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

as proven in the lecture. For the limit to exist when $k \to \infty$, the matrix must be 1×1. Thus, we have proved the forward direction ($\implies$) of the assertion.

(2) *Now we prove the converse.* Note: if $\lambda_i = 1$ and $r_i = 1$, then J_{λ_i} is a 1×1 matrix and $J_1^k = J_1$ for each k. If $|\lambda_i| < 1$, then

$$J_{\lambda_i}^k = \begin{pmatrix} \lambda_i^k & m_1\lambda_i^{k-1} & m_2\lambda_i^{k-2} & \dots & m_{r_i-1}\lambda_i^{k-r_i+1} \\ 0 & \lambda_i^k & m_1\lambda_i^{k-1} & \dots & m_{r_i-2}\lambda_i^{k-r_i+2} \\ 0 & 0 & \lambda_i^k & \dots & m_{r_i-3}\lambda_i^{k-r_i+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i^k \end{pmatrix},$$

where $m_j = \binom{k}{j}$. Now consider $|m_j\lambda_i^{k-j}| = \binom{k}{j}|\lambda_i|^{k-j}$. We claim that $\lim_{k\to\infty}\binom{k}{j}|\lambda_i|^{k-j} = 0$. This follows by treating k as a real variable and applying L'Hospital's rule. Hence as $k \to \infty$, $J_{\lambda_i}^k \to 0$. This proves the first part of (3.1).

Finally, the limit of A^k is non-zero iff the limit of J^k is non-zero, which holds iff the limit of $J_{\lambda_i}^k$ is non-zero *for some i*. From our above result, it follows that the last statement holds iff there is at least one Jordan block with eigenvalue 1.

Problem 3.2, 3.3. Clearly, these two are equivalent so we shall prove them together. As in the solution to (3.1), write $A = SJS^{-1}$, where J is in block Jordan form. Now A^k is bounded iff $J_{\lambda_i}^k$ is bounded for each i.

(1) *Suppose $J_{\lambda_i}^k$ is bounded.* Since its diagonal entries are bounded, we must have $|\lambda_i| \le 1$. Consider the case $|\lambda_i| = 1$. Then we have:

$$\begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}^k = \begin{pmatrix} \lambda_i^k & k\lambda_i^{k-1} & * & \dots & * \\ 0 & \lambda_i^k & k\lambda_i^{k-1} & \dots & * \\ 0 & 0 & \lambda_i^k & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i^k \end{pmatrix}.$$

Now $|k\lambda_i^{k-1}| = k$. So if the matrix remains bounded as k gets large, it must be 1×1, i.e., $r_i = 1$.

(2) *Conversely, suppose $|\lambda_i| < 1$, or $|\lambda_i| = 1$ and $r_i = 1$.* In the second case, there is nothing to prove, since $(1)^k = (1)$ for each k. For the first case, by (3.1) we know that $\lim_{k\to\infty} J_{\lambda_i}^k = 0$, and hence $J_{\lambda_i}^k$ is bounded as k gets large.

Problem 4. Write $A = SJS^{-1}$ yet again, and let J_{λ_i}, r_i be as in the solution to (3.1). Then the characteristic polynomial of A is the same as that of J, which is $p(t) = \prod_{i=1}^m (t-\lambda_i)^{r_i}$. Thus,

$$p(A) = (A - \lambda_1 I)^{r_1}(A - \lambda_2 I)^{r_2}\dots(A - \lambda_m I)^{r_m}.$$

Let us examine a typical term $(A-\lambda_i I)^{r_i}$. We have:

$$A-\lambda_i I = \begin{pmatrix} J_{\lambda_1-\lambda_i} & 0 & \dots & 0 & \dots & 0 \\ 0 & J_{\lambda_2-\lambda_i} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & J_{\lambda_m-\lambda_i} \end{pmatrix}.$$

Now J_0 is an $r_i \times r_i$ matrix. It acts on the elements of the standard basis as follows: $e_1 \mapsto e_2$, $e_2 \mapsto e_3, \dots, e_{r_i-1} \mapsto e_{r_i}$, and $e_{r_i} \mapsto 0$. Hence $J_0^{r_i}$ takes each e_j to 0, and so $J_0^{r_i} = 0$. Hence, $p(A)$ is equal to:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix} \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix} \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix} \dots \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ 0 & * & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where each entry above really refers to an $r_i \times r_j$ block matrix! This product is clearly equal to zero.

Problem 5. Here're the proofs:

(1) $(QZ)(QZ)^* = (QZ)(Z^*Q^*) = Q(ZZ^*)Q^* = QIQ^* = QQ^* = I$. Hence, $(QZ)^*$ is the inverse of QZ, so QZ is unitary.
(2) Since Q is unitary, $QQ^* = I$. Taking the conjugate, we get: $\overline{QQ^*} = \overline{I} = I$. But $\overline{Q^*} = Q^t = (\overline{Q})^*$. Hence, we have $\overline{Q}(\overline{Q})^* = I$, and so $\overline{Q}$ is unitary.
(3) Take the transpose of $QQ^* = I$ to obtain $(Q^*)^t Q^t = I^t = I$. Now, $(Q^*)^t = \overline{Q} = (Q^t)^*$. Hence, $(Q^t)^* Q^t = I$ and so Q^t is unitary.
(4) This follows from (2) and (3).
(5) This follows from (4) since $Q^{-1} = Q^*$.
(6) Since $\langle -, - \rangle$ is the standard inner product, we have $\langle x, y\rangle = y^* x$. This gives:

$$\langle Qx, Qy\rangle = (Qy)^*(Qx) = (y^*Q^*)(Qx) = y^*(Q^*Q)x = y^*x = \langle x, y\rangle .$$

(7) This follows from (6).