Solutions for CS174 Homework 4

Solution 1. Consider a final stable marriage. Because all the males have the same preference ordering, then we can assign each female a unique number k representing her ranking on the lists. Female k has a spouse, and she must be the k-th on his ordering list. This male must have proposed k times. So in total, all males together proposed $\sum_{1\le k\le n} k = n(n+1)/2$ times.

Solution 2. From the definition of expected values, we obtain $\mathrm{E}[X] = \sum_{1\le k\le n} \Pr[X=k]\cdot k$.

Because $\Pr[X\ge k] = \sum_{k\le j\le n} \Pr[X=j]$, we have

$$\begin{aligned}\sum_{1\le k\le n} \Pr[X\ge k] &= \sum_{1\le k\le n}\sum_{k\le j\le n} \Pr[X=j] \\ &= \sum_{1\le j\le n}\sum_{1\le k\le j} \Pr[X=j] \\ &= \sum_{1\le j\le n} \Pr[X=j]\cdot j\end{aligned}$$

Hence, $\mathrm{E}[X] = \sum_{1\le k\le n} \Pr[X\ge k]$.

Solution 3. There are $n!$ possible true permutations, and n^n possible n-permutations. Each permuatation is equaly likely. So the probability that a random n-permutation is a true permutation is $\frac{n!}{n^n} \sim \frac{\sqrt{2\pi n}}{e^n}$.

Solution 4. (a) Think of this as labelling every ball from $1,\dots,m$ with a bin number from $\{1,\dots,n\}$. There are n^m such labellings, and each is equally likely (probability $(1/n)^m$). The number of labellings which have m_1 balls in bin 1 is $\binom{m}{m_1}$ which can be specified by the subset of balls which receive a bin number "1". So the probability is

$$\binom{m}{m_1}\left(\frac{1}{n}\right)^m = \frac{m!}{m_1!\, m_2!}\left(\frac{1}{n}\right)^m$$

(b) If we apply the same argument twice, the number of labellings with m_1 ones is $\binom{m}{m_1}$ and there are $m - m_1 = m_2 + m_3$ balls labelled "not 1". We can further divide the balls labelled "not 1" into groups labelled two and three. The total probability of the specified counts in bins 1, 2, 3 is

$$\binom{m}{m_1}\binom{m-m_1}{m_2}\left(\frac{1}{n}\right)^m = \frac{m!}{m_1!\, m_2!\, m_3!}\left(\frac{1}{n}\right)^m$$

(c) Continuing inductively, The probability that the distribution of m balls into n bins giving $m_1, \ldots, m_n$ balls in bin 1 through bin n, where $m_1 + \cdots + m_n = m$, is

$$\binom{m}{m_1}\binom{m-m_1}{m_2}\cdots\binom{m-m_1\cdots-m_{n-2}}{m_{n-1}}\left(\frac{1}{n}\right)^m = \frac{m!}{m_1!\cdots m_n!}\left(\frac{1}{n}\right)^m$$

Notice that this result contains parts (a) and (b) as special cases. That is, if we substitute $m_n = m_{n-1} = \cdots m_4 = 0$ into this formula, since $0! = 1$, it simplifies to either (a) if $m_3 = 0$ or (b) if $m_3 \neq 0$.