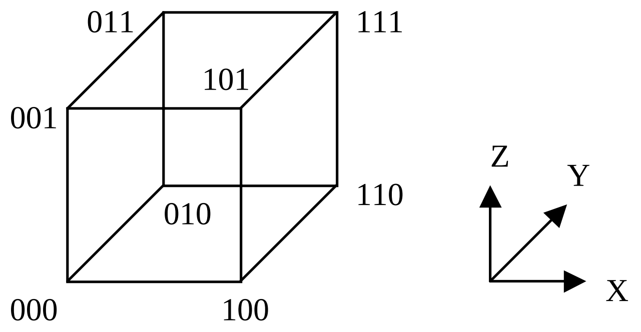
CS174 Lecture 11

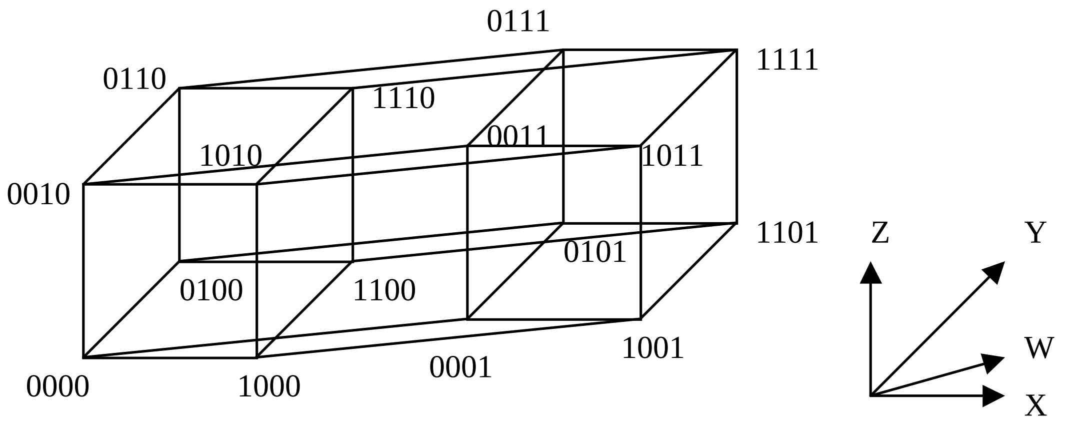
Routing in a Parallel Computer

We study the problem of moving packets around in a parallel computer. In this lecture we will consider parallel computers with hypercube connection networks. The methods we describe are easy to adapt to other connection networks, like butterflys or fat trees. An 8-processor machine is shown below.



The processors are at the vertices of the cube, and edges represent communication links. Each processor is numbered with a 3-bit binary number. Notice that the processor numbers are also X-Y-Z coordinates of the 3D layout of the cube.

A hypercube is a higher-dimensional cube. You can create a hypercube in N+1 dimensions from an N-dimensional by making an extra copy of the N-dimensional cube, and adding an extra bit which is zero in one copy and one in the other. Then join corresponding vertices. E.g. a 4D hypercube can be built from the 3D hypercube this way as shown below.



The processors are number with 4-bit binary numbers which represent the X-Y-Z-W coordinates (aside, this is a valid projection of 4D space, and this is exactly what the 4D cube looks like. Projecting from 4D or N-D into 2D is analogous to 3D projection).

Definitions

Processor number i is wants to send a packet v_i to destination d(i). We assume that the computer runs synchronously, so that all processors try to send their packets at the same time.

Permutation Routing

In permutation routing, every processor tries to send to a different destination. That is, the function d(i) is a permutation.

Oblivious Routing

The route taken by a packet v_i depends only on its destination d(i) and not on any other packet's destination d(j).

Oblivious routing is the most practical method to use for parallel computers. Routing which is not oblivious requires central coordination. But that implies that some processor is able to look at the destinations of many packets in the current time step and compute a good global routing for them. This takes a considerable amount of time and communication. It is certainly not a good idea to use it for routing packets, which is the lowest level operation that the parallel machine performs. Routing must be very fast because all of the high level machine operations rely on it. E.g. some machines even borrow main memory from other machines on the network. So they must access that memory even faster than the simplest computations.

Collisions

A collision occurs when two packets arrive at the same processor at the same time and try to leave via the same link.

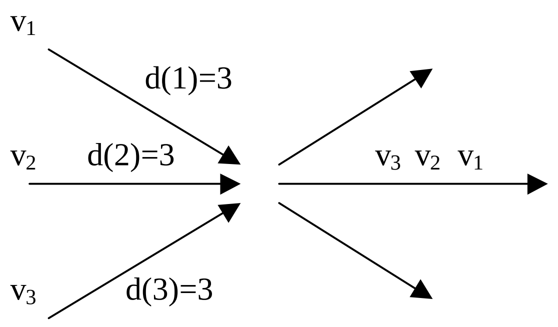
Bit Fixing

Bit fixing is a simple process for routing a single packet obliviously. Take the bit address of the source processor and change one bit at a time to the address of the destination processor. Each time a bit is changed, the packet is forwarded to a neighboring processor.

This works because in a hypercube, each processor i is connected to all the processors j whose bit addresses differ from i in one bit. E.g. processor 1011 is connected to processor 0011, processor 1111, processor 1001 and processor 1010. So changing one bit of the address corresponds to sending the packet from the current node to a neighboring node.

It should be clear that bit fixing is an optimal routing scheme for a single packet. If the source i and destination address j differ by k bits, then the packet must traverse at least k links in the hypercube to get to its destination. Bit fixing gets it there in exactly k steps.

If we have to route many packets though, even if the routing is a permutation routing, bit fixing can cause collisions. In fact, so can any deterministic oblivious routing strategy. To deal with collisions, every processor has a queue and a prioritizing scheme for each incoming packet. If incoming packets try to leave along the same link, they are placed in a queue and sent off in different time steps.



e.g. in the figure above, three packets arrive at the same node and attempt to leave along the same link (we show their destinations as the same which doesn't happen with permutation routing, but what we really mean is that the next node address is the same). If the 3 packets arrive at time step T, the packets are queued, and v_1 leaves in time step T+1, v_2 leaves at T+2, and v_3 leaves at T+3.

Thus routing can still occur with collisions, since all packets eventually get to their destinations, but there may be long delays because of queueing. In fact we have:

Theorem

Any determistic oblivious permutation routing scheme for a parallel machine with N processors each with n outward links requires $\Omega(\sqrt{(N/n)})$ steps.

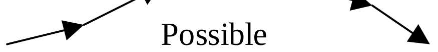
Luckily we can avoid this bad case by using a randomized routing scheme. In fact most permutations cause very few collisions. So the idea is to first route all the packets using a *random* permutation, and then from there to their final destination. That is,

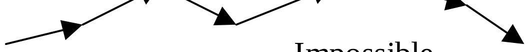
Randomized Routing

1. Choose a random permutation σ of {1,…,N}. Route packet v_i to destination $\sigma(i)$ using bit fixing.

2. Route packet v_i from $\sigma(i)$ to d(i) using bit fixing.

There is an important observation we can make about bit fixing. Namely, during a given phase of the above algorithm (step 1. or step 2.) two packets can come together along a route and then separate, but only once. That is, a pair of routes can look like the left picture but not the right:


Possible


Impossible

To see this, notice that during bit fixing routing, an intermediate address always looks like

$D_1,\ldots,D_k,S_{k+1},\ldots,S_n$

where S_i is a bit of the Source address, and D_j is a bit of the destination address. If two routes collide at the k^{th} step, that means their destination addresses agree in their first k bits and the source addresses agree in their last n-k bits. Let k_0 be the first value of k for which the packets collide. At each time step, we add one more bit of the destination address, which means we increment k. Eventually, the destination bits must disagree (because the destinations are different). Let k_1 be the value of k at which this happens. Then the D_{k1} destination bit is different for the two packets. At that point, the two packets separate. They will never collide again, because all the later intermediate destinations will include the D_{k1} bit.

This observation is the crux of the proof of the following theorem:

Theorem

The delay of a packet v_i is $\leq |S|$ where S is the set of packets whose routes intersect v_i's route.

We won't give a detailed proof. It's enough to notice that whenever the routes of two packets intersect, one of the packets may be delayed by one time step. Once that packet is delayed by one time step at the first shared node, it will flow along the shared route behind the other packet, and will not be delayed any more by that packet.

If the same route intersects other routes, each of them may add a delay of one time step. This happens either (i) because another packet collides with the current packet along a shared route, or (ii) because another packet collides with a packet that is *ahead* of the current packet along a part of the route which is shared by all three. In either case, an extra delay of at most one results.

To get the running time of this scheme, we compute the expected value of the size of the set S above. Define an indicator random variable H_{ij} which is 1 when the routes of packet v_i and packet v_j share at least one edge, and H_{ij} is 0 otherwise. Then by the above theorem, the expected delay of packet v_i is the expected size of S which is

$$E\left[\sum_{j=1}^{N} H_{ij}\right]$$

It's rather difficult to an exact estimate of this quantity because H_{ij} is a complicated condition. It's easier to think about T(e) which is the number of routes that pass through a given edge e. Now suppose the route of packet v_i consists of the edges $(e_1, e_2,\ldots, e_k)$. Then we have

$$\sum_{j=1}^{N} H_{ij} \leq \sum_{l=1}^{k} T(e_l)$$

And so

$$E\left[\sum_{j=1}^{N} H_{ij}\right] \leq E\left[\sum_{l=1}^{k} T(e_l)\right]$$

To use this bound we next compute E[T(e)] which turns out to be ½. To see that, notice that

E[T(e)] = (sum of lengths of all routes)/(total edges in the network)

The sum of lengths of all routes is the expected length of a route times N (the number of all routes). The average length of a route is n/2 because an n-bit source differs from a random destination address is n/2 bits on average. So the sum of route lengths is Nn/2.

The total number of edges in the network is the number of nodes times the number of outbound links, which is Nn. So

$$E[T(e)] = (Nn/2)/(Nn) = ½$$

Then if the path for packet v_i has k edges along it, then

$$E\left[\sum_{j=1}^{N} H_{ij}\right] \le \sum_{l=1}^{k} T(e_l) = \frac{k}{2} \le \frac{n}{2}$$

Now we can apply Chernoff bounds to the probability of there being a substantial number of paths intersecting v_i's path. The Chernoff bound is

$$\Pr\left[\sum H_{ij} > (1+\delta)\mu\right] \le 2^{-\mu\delta}$$

Aside, this is the Chernoff bound for large $\delta > 2e-1$.

We now compute the probability that v_i is delayed at least 3n steps. So we require that $(1+\delta)\mu = 3n$. Notice that we dont actually know what μ is, but we have a bound for it of $\mu \le 0.5n$. It follows that $\mu\delta \ge 2.5n$. Thus the probability v_i is delayed by at least 3n steps is bounded by $2^{-2.5n}$.

This is a bound for the probability that a given packet is delayed more than 3n steps. But we would like to get a bound for the probability that *no* packet gets delayed more than 3n steps. For that, it's enough to use the sum rule for probabilities as a bound. There are $N = 2^n$ routes total, and the probability that none of these takes more than 3n steps is bounded above by

$$2^n 2^{-2.5n} = 2^{-1.5n}$$

So we can make the following assertion:

With probability at least $1 - 2^{-1.5n}$ every packet reaches its destination $\sigma(i)$ in 4n or fewer steps.

The 4n comes from the delay time (3n) plus the time for the bit fixing steps, which is $\le$ n. Notice that all of this applies to just one phase of the algorithm (recall that phase one routes a packet to a random location, and phase two routes from there to the actual destination). So the full algorithm routes all packets to their destinations with high probability in 8n or fewer steps.