Abstract

This paper addresses the problem of non-holonomic motion planning for a class of mobile robots, namely Hilare-like robots. Results from the theory of non-linear systems and optimal control are used to demonstrate the controllability of such systems and to obtain a characterization of the time-optimal paths for such robots in the absence of obstacles. This characterization is applied to find locally time-optimal paths for the system and to obtain a non-holonomic motion planner for such robots navigating through obstacles. Though the ideas presented in this paper focus on a particular class of mobile robots, they should be be applicable to other types of robot systems.

1 Introduction

Earlier approaches to algorithmic robot motion planning were based on the paradigm of the Piano Movers' problem ([Re], [Ca], [SS]). Here the problem of finding an obstacle-avoiding path for a rigid body, from one point to another, can be solved as the equivalent problem of a finding a path for a point in higher-dimensional configuration space avoiding configuration space obstacles. This approach is essentially geometric in nature and ignores the dynamics of the system under consideration. Thus, even though one can find a feasible path in the configuration space of the system, the constraints on the dynamics may preclude the existence of a trajectory for the actual robot. The breakdown of the Piano Movers paradigm is essentially due to the fact that the system under consideration may be non-holonomic, i.e. the dynamical constraints are given by a non-integrable equation. Non-integrability implies that we cannot translate the dynamical constraints into geometrical constraints on the configuration space. As a result, the constraints act on the tangent space (i.e. the space of velocities) of the system and a feasible motion has to obey these constraints.

It appears then that the problem of non-holonomic motion planning presents a degree of complexity more than that of motion planning with configuration space constraints alone. A number of results have emerged in the recent past which appear to make the problem more tractable by the application of concepts from the theories of differential manifolds, Lie Algebras and geometric control theory ([B],[LSi],[LC],[MS],[LSu]). However, most of these results attempt to solve the *FINDPATH* problem, i.e. the problem of finding *some* path (subject to the constraints on hand) if it exists. In particular, there are very few papers which address the general problem of finding optimal paths (e.g. arc-length-optimal or time-optimal paths) for arbitrary systems subject to non-holonomic constraints . The problem becomes even more complicated when the robot has to operate in a configuration space with obstacles.

In this paper, we take first steps towards solving the time-optimal problem for a class of non-holonomic mobile robots, namely Hilare-like robots [GCV]. We use the Maximum Principle of Pontryagin to obtain a (partial) characterization of the time-optimal paths for such robots in the absence of obstacles. This characterization can be used to find locally time-optimal paths for the Hilare system. Based on these ideas, we present a non-holonomic motion planner for Hilare-like robots navigating through obstacles.

2 Related work

The controllability of car-like robots was demonstrated in [L] where the existence of a collision-free trajectory is characterized by the existence of a connected component in the free configuration space. The contructive proof of this result was implemented in a planner in [LTCG]. The case of trailer-like robots is dealt with in [LSi]. Another planner for car-like robots (with or without trailers) is described in [BL] which also contains some of the background for the application of differential geometry concepts to control theory. Other papers which develop the theoretical framework for the study of non-holonomic systems include the ones by Brockett [Br], Li and Canny [LC], Murray and Sastry [MS]. Lafferriere and Sussmann [LSu] give a general strategy for solving the motion planning problem for controllable autonomous systems in the absence of obstacles.

At the other end, there have been several papers dealing with the time-optimal problem outside the above framework. The problem of a holonomic system (e.g. a robot which can be modelled as a point object) is considered in [CRR], where an algorithm is given to find the exact time-optimal trajectory for the case of such a robot moving in the plane among polygonal obstacles subject to bounds on its acceleration. Another focus has been to determine the time-optimal control for a manipulator moving along a predetermined path [BDG,SM]. Other work attempts to characterize time-optimal solutions analytically [Hol, Sch].

3 Hilare-like systems

Figure 1 gives a schematic outline of the dynamics of the Hilare mobile robot. (An overview of the Hilare mobile robot project may be found in [Ch], [GCV]). Hilare has two wheels which are driven independently and four slave caster wheels. The inputs to the system are the accelerations of the two driven wheels. The dynamics of the systems are given by the following set of equations,

$$\dot{x} = \frac{v_1 + v_2}{2} cos\theta \tag{1}$$

$$\dot{y} = \frac{v_1 + v_2}{2} sin\theta \tag{2}$$

$$\dot{\theta} = \frac{v_1 - v_2}{d} \tag{3}$$

$$\dot{v_1} = u_1 \tag{4}$$

$$\dot{v_2} = u_2 \tag{5}$$

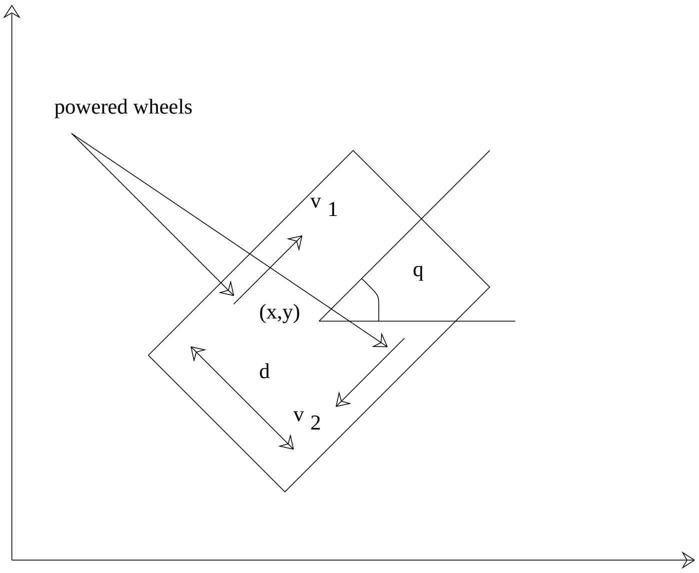

Figure 1: Schematic outline of the Hilare system

$$|u_1(t)| \leq a_{max}, |u_2(t)| \leq a_{max} \tag{6}$$

Here, $(x(t), y(t), \theta(t))$ represent the configuration of the robot at any instant. $v_1(t)$ and $v_2(t)$ are the velocities of the two wheels and $u_1(t)$ and $u_2(t)$ are the respective accelerations, i.e. the inputs to the system, which are bounded by a_{max}. Finally, d is the axle-length as shown in Figure 1.

Similar robots have been described in the literature ([Cr], [KM]). Giralt [G] gives an overview of mobile robots.

4 Reachability for the Hilare system

The **reachability** problem for a dynamical systems can be phrased as follows : given some initial state $\mathbf{x}(t) \in \Re^n$ for the system under consideration, can the system attain any arbitrary final state in the given domain ? The reachability problem can be translated into a statement about the Control Lie Algebra (see [I], [BL], [LC], [LSi]) : any two points can be connected by a path if and only if the dimension of the Control Lie Algebra has maximal dimension. Thus, a system is fully controllable if and only if its Control Lie Algebra has maximal dimension.

Claim: The Control Lie Algebra associated with the Hilare system has maximal dimension (five).

Proof: Let X be the vector field given by

$$X = (\frac{(v_1 + v_2)}{2} \cos\theta, \frac{(v_1 + v_2)}{2} \sin\theta, \frac{(v_1 - v_2)}{d}, 0, 0).$$

We can now use the two vector fields $X_1 = (0,0,0,1,0)$ and $X_2 = (0,0,0,0,1)$ and their Lie brackets to obtain $X_3 = [X, X_1]$, $X_4 = [X, X_2]$, $X_5 = [X_3, X_4]$. It is clear that the fields X, X_1 and X_2

satisfy the constraints of the Hilare system. We can now verify that the five vector fields , X_1 to X_5 are independent by computing their determinant which turns out to be non-zero. It follows then that the Control Lie Algebra has (maximal) dimension 5. ∎

We can therefore conclude that the Hilare system is fully controllable.

5 The maximum principle and Hilare-like systems

In this section, we apply the Pontryagin Maximum Principle (PMP) to the Hilare system, to obtain a partial characterization of the time-optimal path between two points. The Maximum Principle of Pontryagin [P] is a powerful tool from classical optimal control theory which provides necessary conditions for the existence of an optimal control. There are various statements of the principle. A direct consequence of the PMP is the so-called bang-bang principle [AF] Consider a system of the form

$$\dot{\mathbf{x}} = f[\mathbf{x}(t)] + B[\mathbf{x}(t)]\mathbf{u}(t) \tag{7}$$

where $\mathbf{x}(t) \in \Re^n$, $\mathbf{u}(t) \in \Re^m$, $f[\mathbf{x}(t)] \in \Re^n$ and $B[\mathbf{x}(t)] \in \Re^{n\times m}$, where $|u_j(t)| \leq 1$. Under certain continuity assumptions (see [P] or [AF]), we have,

Lemma 1 *(The Bang-Bang principle) Let $\mathbf{u}^*(t)$ be a time-optimal control for the system under consideration. Let $\mathbf{x}^*(t)$ and $\psi^*(t)$ be the corresponding state trajectory and costate. If the problem is* **normal** *then the components $\mathbf{u_j}^*(t)$, $i = 1, \ldots, m$ must be defined by the relation*

$$\mathbf{u_j}^*(t) = -\mathbf{sgn}\{\sum_{i=1}^{n} b_{ij}(\mathbf{x}^*(t), t)\psi_i{}^*(t)\} \quad j = 1, \ldots, m \tag{8}$$

for $t \in [t_0, T^*]$, where T^* is the minimum time.

The Bang-Bang Principle says that if the system under consideration is *normal* [AF, p. 381], and an optimal control exists then it must necessarily lie on one of the corners of the hypercube of side 2 a_{max} in $\Re^m$. at all instances till the goal state is achieved. However it is possible that the system is *singular* [AF, p. 381], i.e. there is at least one interval $(t_1, t_2]$ in $[0, T^*]$ such that

$$\sum_{i=1}^{n} b_{ij}(\mathbf{x}^*(t), t)\psi_i{}^*(t) = 0 \qquad \forall t \in (t_1, t_2] \tag{9}$$

In this case, the Bang-Bang Principle does not apply. We will now show that the Hilare system is not singular. We can then conclude that the optimal control always lies on one of the corners of the hypercube.

Claim: The Hilare system given by equations (1)-(6) is not singular

Proof: Assume that the Hilare system is singular. We can form the Hamiltonian corresponding to this system :

$$H = 1 + \psi_1 \frac{v_1 + v_2}{2} \cos\theta + \psi_2 \frac{v_1 + v_2}{2} \sin\theta + \psi_3 \frac{v_1 - v_2}{d} + \psi_4 u_1 + \psi_5 u_2 \tag{10}$$

We will consider the case where the first control u_1 is singular and the second is extremal. The other cases, where u_2 alone is singular and both controls are singular can be dealt with similarly. For u_1 to be singular we have, [AF, pp. 481-493],

$$\dot{\psi}_4 = -\frac{\partial H}{\partial v_1} = 0$$

This implies,

$$\frac{\psi_1}{2}\cos\theta + \frac{\psi_2}{2}\sin\theta + \frac{\psi_3}{d} = 0 \tag{11}$$

or

$$\psi_3 = -d\frac{\psi_1}{2}\cos\theta - d\frac{\psi_2}{2}\sin\theta \tag{12}$$

Hence

$$\dot{\psi}_3 = d\frac{\psi_1}{2}\sin\theta\cdot\dot{\theta} - d\frac{\psi_2}{2}\cos\theta\cdot\dot{\theta} \tag{13}$$

Putting $\dot{\theta} = \frac{v_1 - v_2}{d}$ and rearranging we get,

$$\dot{\psi}_3 = (\psi_1\sin\theta - \psi_2\cos\theta)\frac{(v_1 - v_2)}{2} \tag{14}$$

But, by definition,

$$\dot{\psi}_3 = -\frac{\partial H}{\partial \theta}$$

Therefore, taking the partial derivative and rearranging we get,

$$\dot{\psi}_3 = (\psi_1\sin\theta - \psi_2\cos\theta)\frac{(v_1 + v_2)}{2} \tag{15}$$

Hence either

1. $v_1 - v_2 = v_1 = v_2$ or
2. $(\psi_1\sin\theta - \psi_2\cos\theta) = 0$

The first case implies $v_2 = 0$ for all $t \in (t_1, t_2]$. But since u_2 is extremal this is not possible. The second case implies

$$\tan\theta = \frac{\psi_2}{\psi_1}$$

i.e. the motion is along a straight line. But it is clear that the time-optimal motion along a straight line is obtained when both controls are extremal and have the same sign (this can be shown using the PMP for example). Thus neither case can hold and therefore u_1 cannot be singular. The other cases are handled similarly. We thus conclude that the Hilare system is not singular.

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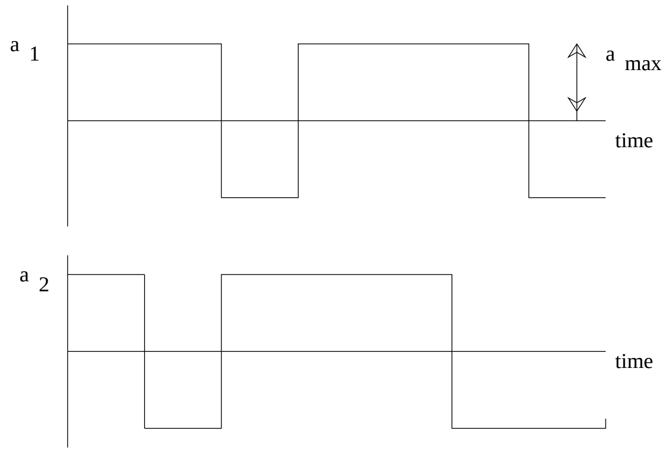

Figure 2: General form of the optimal trajectory

6 The form of time-optimal trajectories

In the previous section, we showed that for the Hilare system, the optimal control always lies on the corners of the square of size 2 a_{max} centered at the origin of the control space. In other words, on an optimal trajectory the accelerations for the two wheels are $\pm a_{max}$. Thus the time-optimal trajectory in the absence of obstacles is generated by input accelerations with some number of switches as shown in Fig. 2. It is not clear *a priori* how many switches are required and this remains an important open problem in the characterization of the time-optimal trajectory. We now study the forms of the curves generated by these input values. We get four cases corresponding to the four corners of the square in control space. However, the opposing corners give rise to cases which are symmetric and so we need to deal with only two cases.

6.1 The clothoid and the anti-clothoid

When the two accelerations u_1 and u_2 have opposite signs, the trajectory obtained has the property

$$\frac{v_1(t) + v_2(t)}{2} = \nu = constant \tag{16}$$

I.e. the trajectory has the property of constant linear velocity. The corresponding curve $(x(t), y(t))$ is called a **clothoid** or **Euler's spiral** (Figure 3). It follows that the arc length is given by νt, a linear function of time. Unfortunately, the corresponding expressions for x and y are not integrable. When both start velocities $v_1(0)$ and $v_2(0)$ are zero, the clothoid degenerates to a pure rotation of the robot about its axle center. There are two cases depending on whether the acceleration for the first or second wheel is positive (the other being negative). We denote the first case ($a_1 = u_1$ positive) Clothoid+ (or C+) and the second case ($a_2 = u_2$ positive) Clothoid- (or C-).

When the two accelerations u_1 and u_2 have the same sign the trajectory obtained has the property

$$\dot{\theta} = \frac{v_1(t) - v_2(t)}{d} = \kappa = constant \tag{17}$$

Thus, we get a constant angular velocity trajectory and the corresponding curve can be very aptly called an **anti-clothoid** or a **dual clothoid**. The angular rotation is given by κt. and we can solve

Figure 3: The clothoid

in this case for x and y exactly. When both start velocities are zero the anti-clothoid degenerates to a pure translation in the initial direction. As before, there are two cases, the first corresponding to both accelerations positive is denoted A+ and the other (both accelerations negative) A-. In light of the above discussion, it is clear that in some sense the clothoid is the faster curve for rotation-intensive motion whereas the anti-clothoid is more suited for translation-intensive motion.

7 A non-holonomic motion planner for Hilare

We now present a non-holonomic motion planner for Hilare. The motion planner finds trajectories which are provably locally time-optimal in the homotopy class of the given trajectory with or without the presence of obstacles. As mentioned earlier, unless we have an upper bound on the number of switches in the optimal trajectory between any two points, we cannot achieve *provably* globally time-optimal paths in the absence of obstacles, or *provably* locally time-optimal paths when obstacles are present. In other words, we may achieve the globally optimal trajectory, but would not be able to prove it to be so. In practical terms this is not very important, since from the view-point of computation, we would be able to investigate trajectories with only a (predetermined) bounded number of switches.

The following gives an overview of the modules of the Hilare motion planner.

- A geometric motion planner, which ignores the non-holonomic constraints, and returns an obstacle-avoiding trajectory
- An iterative planner which uses a sub-division scheme (which is guaranteed to converge) to refine the trajectory found in the first step into a trajectory which satisfies the non-holonomic constraints.

- A non-linear optimizer which takes a given trajectory and returns one which is locally time-optimal within its homotopy class.

We now discuss the various parts of the planner in greater detail.

7.1 The geometric planner

The Hilare planner takes as input an initial point in state space, i.e. $(x(t_0), y(t_0), \theta(t_0), v_1(t_0), v_2(t_0))$, a final point $(x(t_f), y(t_f), \theta(t_f), v_1(t_f), v_2(t_f))$, and a description of the polygonal obstacles. The first stage of the planner finds a collision-free path for the robot without taking into account the non-holonomic constraints. This is essentially the problem of motion planning for a (possibly non-convex) polygon amidst polygonal obstacles. Avnaim et al. [ABF] give an algorithm which computes an exact boundary description of the free space in time $O(n^3 \log n)$ and a motion in time $O(n^3)$. Here n is the number of vertices of the polygonal obstacles. It is entirely possible, of course, to use another algorithm which returns a path which lies wholly in free space.

7.2 The iterative planner and optimizer

The next step is to convert the trajectory obtained in the first step into one which satisfies the non-holonomic constraints. This is done using a sub-division scheme as follows : the planner decides on the form of the trajectory (e.g. C+A+C-A-C+) and then calls the non-linear optimizing routine to find a locally optimal trajectory in this class. It then uses a collision detecting routine to determine if the trajectory obtained is obstacle-avoiding or not. If yes, the process stops and the planner returns the trajectory. If not, the planner divides the original trajectory into two pieces and the applies the same process. The subdivision continues recursively till we find a trajectory which is collision-free. It can be shown that this process will converge after a *finite* number of steps. Further within its class, the trajectory found will be locally time-optimal.

In actual practice, it seems more efficient to commence with a finer subdivision of the original collision-free trajectory and then apply the above process. This tends to avoid a large number of expensive calls to the collision detector.

8 Conclusions and open problems

In this paper we presented an application of theoretical results from various branches of non-linear and optimal control theory and algorithmic robotics to the problem of motion planning for a class of robots. Though the focus of this paper was Hilare-like robots, a similar approach should yield planners for other types of mobile robot systems.

The two most important open problem associated with the above system are

1. Giving an upper bound on the number of switches in the optimal trajectory between two states of the system in the absence of obstacles. This will enable us to find trajectories which are provably time-optimal.

2. Providing an efficient algorithm for synthesizing the time-optimal trajectory, once the first objective has been achieved.

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