Problem Set 4 for CS 170

Problem 1 [Robustness of Dijkstra]

Suppose that we are given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, all other edge weights are nonnegative, and there are no negative-weight cycles. Argue that Dijkstra's algorithm correctly finds shortest paths from s in this graph.

Problem 2 [Existence of Fast Relaxation Sequence]

Let $G = (V, E)$ be a weighted, directed graph that contains no negative-weight cycles. Let $s \in V$ be the source vertex, and let $\delta(s, v)$ denote the shortest-path distance from s to v, for all $v \in V$. Prove that there exists a sequence of $|V| - 1$ relaxation steps that produces $\texttt{dist}[v] = \delta(s, v)$ for all $v \in V$.

Problem 3 [Second Best Tree]

Let $G(V, E)$ be an undirected, connected, weighted graph, such that the weights on all the edges are distinct.

(a) Prove that there is a unique minimum spanning tree in $G(V, E)$.

The goal of the following three parts is to design an efficient algorithm to compute the second-best spanning tree in $G(V, E)$ (i.e., a spanning tree which has the second lowest total weight among all possible spanning trees of $G(V, E)$).

(b) Let T be a minimum spanning tree of $G(V, E)$. Prove that there exist edges $\{x, y\} \in T$ and $\{u, v\} \notin T$ such that $T - \{x, y\} \cup \{u, v\}$ is a second-best spanning tree of $G(V, E)$. Show that this property gives an $O(|E|^2)$ algorithm for computing a second-best spanning tree.

Next we will show how to speed up this algorithm by more quickly figuring out which edges to change in the minimum spanning tree T to get the second-best spanning tree:

(c) Let T be a minimum spanning tree of $G(V, E)$, and for any two vertices $u, v \in V$, let $M(u, v)$ be an edge of maximum weight on the unique path between u and v in T. Describe an $O(V^2)$ time algorithm that on input T, computes $M(u, v)$ for all $u, v \in V$.

(d) Based on the previous part, give the most efficient algorithm you can for finding a second-best spanning tree of $G(V, E)$.