**CS 174 Homework Assignment 1** (due Wednesday, Feb. 6)

1. Consider a discrete random variable $X$ whose range is the positive integers and whose probability mass function is $P(X = x) = Cx^{-2}$, where $C$ is a generic constant such that the probabilities sum to one. Does the expected value $\mathrm{E}[X]$ of this random variable exist? What about the case for which $P(X = x) = Cx^{-3}$?

2. (a) Design a sequence of discrete random variables $X_n$ each of whose probability distributions are supported on two points, such that $\mathrm{E}[X_n] = 1$ and $\mathrm{Var}[X_n] \to \infty$.
   (b) Design a sequence of discrete random variables $X_n$ each of whose probability distributions are supported on two points, such that $\mathrm{E}[X_n] = 1$ and $\mathrm{Var}[X_n] \to 1$.

3. A system is called a "$k$ out of $n$" system if it contains $n$ components and it works whenever $k$ or more of these components are working. Suppose that each component is working with probability $p$, independently of the other components, and let $X_c$ be the indicator function of the event that component $c$ is working. Find, in terms of the $X_c$ the indicator function of the event that the system works, and deduce the *reliability* of the system; i.e., the probability that the system works.

4. Answer the following questions about the balls and bins model:
   (a) In the case $m = n$ ($n$ balls, $n$ bins), what is the expected number of bins that contain *exactly one* ball, as $n \to \infty$?
   (b) What is the expected number of empty bins if the number of balls $m$ is $2n$?
   (c) Let $m = n$, and let $Y_k$ be the number of bins that contain at least $k$ balls. If you follow through the calculations on pages 3-4 of Note 1, you will see that, for all $n, k$, $\mathrm{E}Y_k \le \frac{1}{1-\mathrm{e}/k} n (\frac{\mathrm{e}}{k})^k$. With $n = 1$ million, use this to obtain upper bounds on the probability that any bin contains more than $k$ balls, for $k = 3$ through 16.

5. Consider the following algorithm for finding the *median* of a set $S$, where the number of elements $n$ of $S$ is assumed even, and where the initial call is $\textsc{FindRank}(S, n/2)$:

   $\textsc{FindRank}(S, k)$

   1. Pick an element $s$ uniformly at random from $S$.
   2. Split $S$ into two pieces: $S_1 = \{x \in S : x < s\}$ and $S_2 = \{x \in S : x > s\}$.
   3. If $|S_1| = k - 1$ then output $s$.
      If $|S_1| > k - 1$ then output $\textsc{FindRank}(S_1, k)$.
      If $|S_1| < k - 1$ then output $\textsc{FindRank}(S_2, k - |S_1| - 1)$.

   Show that the expected number of comparisons made by $\textsc{FindRank}(S, n/2)$ is at most $4n$.