CS 174 Homework Assignment 2 (due Wednesday, Feb. 13)

1. Define the *covariance* of random variables X and Y as follows:

$$\mathrm{Cov}(X,Y) = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])].$$

Show the following:

$$\mathrm{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]E[Y].$$

Show also that if two random variables are independent then their covariance is zero.

2. Let X and Y be Bernoulli random variables with $P(X = 1) = P(Y = 1) = \frac{1}{2}$. Show that the random variables $(X + Y)$ and $|X - Y|$ have zero covariance, but that they are not independent.

3. For a given pair of integers i and j, what is the probability that i and j are adjacent in a random permutation of the integers 1 to n?

4. You are given a biased coin which comes up heads with probability p and tails with probability $1 - p$.

   (a) What is the expected number of times you need to toss the coin before the first head appears (including the head itself)?

   (b) What is the variance of the random variable in part (a)?

   (c) Suppose we want to use this biased coin to generate truly random bits. We can use a trick due to von Neumann, namely, toss the coin twice and do the following: if the two flips are HT, output 0; if the two flips are TH, output 1; otherwise, fail (with no output).

   (d) Why is this procedure unbiased?

   (e) What is the expected number of tosses we will need to generate a single random bit? What is the variance?

5. Consider a random graph $G = (V, E)$ on n vertices, where edges are chosen independently with probability $P(n)$. Let $P(n) = (\ln n + c)/n$ for some $c \in \Re$.

   (a) Show that the probability that one or more vertices is *isolated* (that it has no neighbors) is at most e^{-c}.

   (b) Show that the probability that one or more vertices is isolated is at least $e^{-c} - \frac{1}{2}e^{-2c}$.