**CS 174 Homework Assignment 5** (due Wednesday, March 13)

1. Consider a Poisson random variable $X$ with parameter $\lambda = 1$. Calculate upper bounds on the probability $P(X \geq 3)$ using (a) the Markov bound, and (b) the Chebyshev bound. Also, (c) compute the exact probability.

2. (a) Find a (simple) random variable such that the Markov bound is an equality (i.e., a random variable that *attains* the Markov bound). (b) Find a (simple) random variable that attains the Chebyshev bound.

3. For a positive integer $k$, the problem MAX-$k$SAT is defined exactly as for MAXSAT except that the input formula $\phi$ must have precisely $k$ literals in every clause. Show that, for a MAX-$k$SAT formula $\phi$, there always exists an assignment that satisfies at least seven-eighths of the clauses of $\phi$.

4. A *dominating set* of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V - U$ has at least one neighbor in $U$. Let $G = (V, E)$ be a graph on $n$ vertices, with minimum degree $\delta > 1$. We wish to prove that $G$ must have a dominating set of at most $n\frac{1+\ln(\delta+1)}{\delta+1}$ vertices.

   (a) Given a fixed $p \in [0, 1]$, pick vertices from $V$, randomly and independently with probability $p$. Let $X$ denote the (random) set of picked vertices. What is the expected value of $|X|$?

   (b) Let $Y$ be the random set of all vertices in $V - X$ that do not have any neighbor in $X$. (Note that $Y$ depends on $X$). Compute an upper bound on $P(v \in Y)$, for any fixed $v \in V$.

   (c) Using your results from (a) and (b), show that there is at least one choice of $X \subseteq V$ such that $|X| + |Y| \leq np + n(1-p)^{\delta+1}$. Define the set $U = X \cup Y$. Argue that $U$ is a dominating set of $G$ whose cardinality is at most $np + n(1-p)^{\delta+1}$.

   (d) The upper bound established in (d) holds for any $p$. Take the derivative with respect to $p$ and compute the value of $p$ that yields the tightest upper bound. (Use the bound $1 - p \leq e^{-p}$ to make the calculus easier). Substitute this value of $p$ back into the bound on $|U|$ to obtain the desired result.

   This line of argument is an example of the *alteration* method. One starts with a random set ($X$ in this case) that "almost" works, and "alters" it (adds the vertices in $Y$) to get a set that does work.