CS 174 Quiz 2
April 3, 2002

1. [10 pts] Consider a nonnegative random variable X with mean equal to one and variance equal to one. Consider also a sequence of independent, nonnegative random variables $\{X_i\}$, where each X_i has mean equal to one and variance equal to one, and let $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) Calculate an upper bound on $\Pr(X \geq 3)$ using Markov's inequality.

   (b) Calculate an upper bound on $\Pr(X \geq 3)$ using Chebyshev's inequality.

   (c) Compute the mean and variance of M_n.

   (d) Calculate an upper bound on $\Pr(M_n \geq 3)$ using Markov's inequality.

   (e) Calculate an upper bound on $\Pr(M_n \geq 3)$ using Chebyshev's inequality.

   (f) What can you conclude about how $\Pr(M_n \geq 3)$ behaves as $n \to \infty$ from these results?

   (g) Now suppose that the $\{X_i\}$ are no longer independent, but that all of their covariances $\mathrm{Cov}(X_i, X_j)$ are of magnitude $O(1/n)$. Does the conclusion in (f) still hold?

2. [10 pts] Recall the random graph model $\mathcal{G}_{n,p}$:

   - Given a set E, initially empty, and a vertex set $V = \{1, 2, \ldots, n\}$
   - For each of the edges $e = \{i, j\}$, independently flip a coin with heads probability p, and if the coin comes up heads, add e to E
   - Output $G = (V, E)$

   Pick a graph G randomly from $\mathcal{G}_{n,p}$ and color each vertex of G with one of two colors, uniformly at random.

   (a) For a given triple of vertices in G, what is the probability that these vertices are connected; i.e., that we obtain a triangle?

   (b) What is the expected number of triangles in G?

   (c) What is the expected number of monochromatic triangles?

   (d) How do we set p as a function of n so that the expected number of monochromatic triangles is approximately equal to one?

3. [10 pts] Let the Boolean variables $(x_1, x_2, \ldots, x_n)$ be set randomly and independently to `true` or `false` with probability 1/2.

   (a) What is the expected number of clauses that are satisfied in the conjunctive normal form (CNF) formula $\phi_C = (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3) \wedge (x_1 \vee x_3 \vee x_5)$?

   (b) Now consider the *disjunctive normal form (DNF)* formula $\phi_D = (x_2 \wedge \bar{x}_3) \vee (\bar{x}_1 \wedge x_3) \vee (x_1 \wedge x_3 \wedge x_5)$. Refer to each expression in parentheses (each conjunct) as a "term." What is the expected number of terms that are satisfied in this formula?

   (c) Give an upper bound on the probability that ϕ_D evaluates to `true`. For what kinds of DNF formulas is such an upper bound tight?

   (d) Now consider the class of "k-DNF Boolean formulas." These are DNF formulas (disjunctions of conjunctions), where each "term" (i.e., each conjunct) is limited to no more than k literals. For example, $(x_1 \wedge \bar{x}_2 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4)$ is a 3-DNF formula with two terms.

   Let ϕ be a particular fixed k-DNF formula with m terms. Let X be a random variable that counts the number of terms in ϕ that evaluate to `true`. How big can k be, as a function of m, such that E[X] remains greater than or equal to one?

4. [10 pts] Suppose that each student at Secure University belongs to one or more study groups, where each study group has exactly m students in it. To enter into the study hall, the group needs to know a secret password, but each individual student is allowed to know only the first half of the password or the second half of the password, but not both halves. We want to ensure that each study group has at least one person who knows the first half of the password and one person who knows the second half of the password, so that the group can put the halves together and get into the study hall. Assuming that the number of study groups is no greater than 2^{m-1}, show that we can find an assignment (of halves of the secret to students) so that our goal can be achieved.

5. [8 pts] Let X be a random variable denoting the outcome of a biased coin toss, with probability of heads p. Let $H(X) = -p \log p - (1-p)\log(1-p)$ denote the entropy of X. Consider a sequence of n independent tosses of the coin.

   (a) Suppose that there are k heads in the sequence. What is the probability of the sequence?

(b) Suppose that there are exactly np heads in the sequence (assuming that n and p are chosen such that np is an integer). What is the probability of the sequence?

(c) Rewrite the probability in (b) in terms of the entropy $H(X)$.

(d) Continuing to assume that there are exactly np heads in the sequence, is the sequence necessarily in the typical set $A_\epsilon^{(n)}$ for $\epsilon = .01$?