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Disclaimer: *These notes have not been subjected to the usual scrutiny for formal publications. They are to be used only for the class.*

Outline:

1. Notations of Convergence
2. Labeled and Unlabeled Balls and Bins
3. Derangements

1 Notations of Convergence

Suppose two series $\{x_n\}$ and $\{y_n\}$, where $n = 1,2,3,\ldots$. Then I can define the following four symbols in terms of limits (lim).

$$\begin{aligned} x_n \to x &\Leftrightarrow \lim_{n\to\infty} x_n = x \\ x_n \sim y_n &\Leftrightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = 1 \\ x_n = O(y_n) &\Leftrightarrow |\frac{x_n}{y_n}| < C \text{ for sufficiently large } n \\ x_n = o(y_n) &\Leftrightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = 0 \end{aligned}$$

Probably most of you are already familiar with these notations; this is just to bring everyone to the same baseline to avoid confusion.

Question: Does $x_n \sim y_n$ imply $\exp(x_n) \sim \exp(y_n)$? In general, the answer is no. One counter-example is to let $x_n = n^2 + n$ and $y_n = n^2$. However, let me ask a slightly different question: under what condition can we "exponentiate both sides"?

Claim: $x_n - y_n \to 0 \Rightarrow \exp(x_n) \sim \exp(y_n)$

Proof: Because the function $\exp(\cdot)$ is continuous,

$$\lim_{n\to\infty} \exp(x_n - y_n) = \exp(\lim_{n\to\infty}(x_n - y_n)) = \exp(0) = 1$$

Hence, $\frac{\exp(x_n)}{\exp(y_n)} \to 1$.

Using this claim, Fact 1 in lecture notes 1 turns out to be quite apparent: show that if t is a function of n, as long as $t = o(\sqrt{n})$,

$$(1+\frac{t}{n})^n \sim e^t$$

Proof:By the claim, all we need to do is to verify that the logarithm of both sides differ by a vanishing amount, that is,

$$n\ln(1+\frac{t}{n}) - t = n\frac{t}{n} + O(n\cdot\frac{t^2}{n^2}) - t = O(\frac{t^2}{n}) \to 0$$

because of the condition that $t = o(\sqrt{n})$. Generally, the handy claim above should make it easier to tackle your homework problems.

2 Labeled and Unlabeled Balls and Bins

In homework 1, you had following question concerning balls and bins: compute the probability that bin 1 contains exactly 1 ball. Let there be m bins and n balls; some of you say it's $\frac{n(m-1)^{n-1}}{m^n}$; some of you say it's $\frac{(m-1)^{n-1}}{m^n}$. Which is correct?

Defenders of the second answer say that one ball falls into bin 1 with probability $\frac{1}{m}$; and all the other balls fall into other bins with probability $(1-\frac{1}{m})^{n-1}$. Multiply these two terms give the second answer.

However, this is wrong by a subtle point. Because the balls are *distinct*, there can be n distinct balls falling into bin 1, hence the factor of n.

Now, one might still say that the probability has nothing to do whether the balls are distinct or not. Even if the balls are not distinct, we should still get the same probability.

Upon closer look, this amounts to a different sample space. Previously, the sample space Ω is all the ways that the balls fall inside the bins. (and $|\Omega| = m^n$.) Now, without labeling, the balls are indistinguishable from each other. Therefore, a sample point in the new sample space Ω' is of the form $(n_1,\ldots,n_m)$ where n_i denotes the number of balls in bin i (since they are all the same, only the count matters.) By a counting method which I won't go into here, the size of the sample space is reduced to $\binom{m+n-1}{m}$.

So, you can see that the sample space is smaller. Moreover, the probability distribution is changed, too. For Ω, the probability is evenly distributed to all sample points. However, for a sample point $(n_1,\ldots,n_m)$ in Ω', the probability is $\binom{n}{n_1 n_2 \ldots n_m}/m^n$ where $\binom{n}{n_1 n_2 \ldots n_m} = \frac{n!}{n_1! n_2! \ldots n_m!}$, denotes the number of ways to throw n balls to give rise to the configuration of $(n_1,\ldots,n_m)$ number of balls in the bins. These numbers are called "multi-nomial coefficients" and are extensions to the binomial coefficients. Similar to the binomial sum, they sum up to m^n, which is used to normalize the probability for them to sum up to 1.

In this setting, we compute the same probability that bin 1 contains exactly 1 ball, which is the sum of probabilities of all the sample points that satisfy this condition:

$$\begin{aligned}
\sum_{n_2,\ldots,n_m} \binom{n}{1\, n_2 \ldots n_m}/m^n &= \frac{n \sum_{n_2,\ldots,n_m} \binom{n-1}{n_2 \ldots n_m}}{m^n} \\
&\quad \text{easy to verify by the definition of multinomial coefficients} \\
&= \frac{n(m-1)^{n-1}}{m^n} \\
&\quad \text{by the multinomial sum of m-1 bins and n-1 balls}
\end{aligned}$$

The answer is still the same, but the derivation is a lot more involved. Why? The difference lies in the choice of the sample space. A good choice of the sample space saves a lot of toil for this problem. For more complicated problem, a careful construction of the sample space may be the key to solve the problem.

3 Derangements

Let D_n be the number of permutations without a 1-cycle(i.e., no element stays in it original position after being permuted). Let E_i be the event that element i stays in its position. Then we are interested in computing $D_n = |\overline{E_1} \cap \overline{E_2} ... \overline{E_n}| = |\overline{E_1 \cup E_2 ... \cup E_n}|$, where $|\cdot|$ denotes the cardinality of the set.

This looks liable to Principle of Inclusion and Exclusion (PIE). To use PIE, let's play around with $|E_i| = (n-1)!$, $|E_i \cap E_j| = (n-2)!$, $|E_i \cap E_j \cap E_k| = (n-3)!$, ... Therefore,

$$S_k = \# \text{ of } k\ E_i\text{'s} \cdot |\text{intersection of } k\ E_i\text{'s}| = \binom{n}{k}(n-k)! = \frac{n!}{k!}$$

Therefore, the answer is:

$$\begin{aligned} D_n &= n! - \frac{n!}{1!} + \frac{n!}{2!} - ... + (-1)^n \frac{n!}{n!} \\ &= n![1 - \frac{1}{1!} + \frac{1}{2!} - ... + (-1)^n \frac{1}{n!}] \\ &= n![\frac{1}{e} - (\frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + ...)] \\ & \quad \text{by the Taylor series expansion of } \exp(-1) \\ &= \frac{n!}{e} - n![...] \\ & \quad \text{the term n![...] can be easily bound by } \tfrac{1}{2} \text{ for } n > 3 \text{, using telescoping} \\ &= [\frac{n!}{e}] \end{aligned}$$

where $[\cdot]$ denotes the nearest integer.

Using derangments, here is another way of solving the problem presented in lecture notes 3:

$$\begin{aligned} q_k &= \Pr[\text{random permutation contains exactly } k \text{ 1-cycles}] \\ &= \frac{\#\text{ of permutations that has } k \text{ fixed positions}}{\text{total } \# \text{ of permutations}} \\ &= \frac{\binom{n}{k} D_{n-k}}{n!} \\ &= \binom{n}{k}[\frac{(n-k)!}{e}]/n! = \frac{[\frac{n-k)!}{e}]}{k!(n-k)!} \\ & \quad \text{when } n \text{ is large and } k \text{ is fixed} \\ &\approx \frac{(n-k)!/e}{k!(n-k)!} = \frac{1}{e \cdot k!} \end{aligned}$$

With the tool of derangments and a little derivation, this leads to the same conclusion that the distribution converges to a Poisson as n grows large. Also, the reason why this approximation fails for $k = n-1$ is attributed to the approximation of $[(n-k)!/e]$, which only holds when $n-k$ is large, i.e., > 3.

(Aside: For a proof of the Bonferroni inequality in its general form as appeared in lecture notes 3, see, for example, J. Galambos, I. Simonelli, "Bonferroni-type Inequalities with Applications", pp. 14-15)