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**Disclaimer**: *These notes have not been subjected to the usual scrutiny for formal publications. They are to be used only for the class.*
**Outline**:

1. Recurrent state
2. Equivalence between Markov Chains and Random Walk on Graphs

# 1 Recurrent state

Consider a simple random on the line as follows:

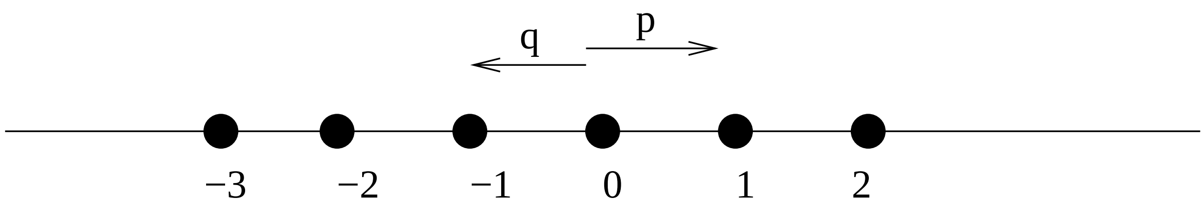


Figure 1: Simple 1-D Random Walk

The particle goes to the right with probability $p$ and to the left with probability $q$. Let $p(2n)$ be the probability that the particle returns to the origin after $2n$ steps. Then, it turns out that

$$p(2n) = \binom{2n}{n} p^n q^n$$

To see this, think of the analog of coin flips with head probability $p$. Now, I claim that

$$p(2n) \sim \frac{(4pq)^n}{\sqrt{\pi n}}$$

It is a good exercise for you to verify the above claim. Hint: by Stirling, $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$. Note that $p+q=1$, therefore, $pq = p(1-p)$ is zero at $p=0$ and $p=1$ and is maximized at $p=\frac{1}{2}$, at which point $pq=\frac{1}{4}$. Therefore, when $p=q=\frac{1}{2}$,

$$p(2n) \sim \frac{1}{\sqrt{\pi n}}$$

Contrast this with an "asymmetric" walk where $p \neq \frac{1}{2}$, in which case $p(2n)$ follows a different kind of asymptotic fall-off

$$p(2n) \sim \frac{\lambda^n}{\sqrt{\pi n}}$$

where $\lambda < 1$. These results suggest that when the walk is asymmetric, it is far less likely for the particle to return to the origin (the $p(2n)$s follow an exponential decay as opposed to a $O(\frac{1}{\sqrt{n}})$ fall-off). This statement agrees with intuition that when the walk is biased in one direction, the particle is likely to wander off far away after many steps.

In fact, it turns out that the particle returns to origin infinitely often if and only if $p = \frac{1}{2}$. To prove this, one would need precise notion of "first return": let $f(n)$ be the probability that the particle returns to the origin *for the first time* after $n$ steps. (In contrast, in the definition of $p(2n)$, the particle may have already returned to the origin before the $2n$'th step). Then it can be shown that [1]:

$$\sum_{n=1}^{\infty} f(n) = 1 - |p - q|$$

and

$$\sum_{n=1}^{\infty} n f(n) = \infty$$

The first expression tells us that the particle returns to the origin with probability 1. However, the second expression denotes the "expected return time" and it is infinity. Namely, the particle is destined to return to the origin but only after an infinite amount of time. (an apparent paradox to think about.) To draw an analog, consider the analysis of the gambling technique called "martingale" [1]. When you play a "martingale", you bet \$1, 2, 4, ... successively until the first win, at which point you win \$1 over the previous losses. With probability 1, you win \$1 for the whole game. However, the expected loss before the win (namely the capital you must bring with you) is

$$\begin{aligned}
E[\text{loss}] &= \sum_{l=1}^{\infty} \text{cumulative loss till round } i \times \Pr[\text{losing the first } i \text{ rounds but winning } i+1 \text{ round}] \\
&= \sum_{l=1}^{\infty} (2^l - 1) \frac{1}{2^{l+1}} \\
&= \infty
\end{aligned}$$

## 2 Equivalence between Markov Chains and Random Walk on Graphs

In class, we have touched upon the connection between a random walk on a graph and a Markov chain. In fact, as Prof Jordan mentioned in class, those two entities are the same thing, namely, one problem can be reduced to the other.

Let's first convert a random walk on a graph into a Markov chain. Consider the following graph:

[1] this term is also used in probability theory to denote a different but somewhat related notion

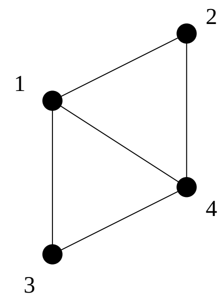


Figure 2: Simple Graph

Construct a matrix $P$ whose $i,j$ entry is the probability of going from node $i$ to node $j$:

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Suppose further that the start state is on node 2, let $p_0 = (0,1,0,0)$. Then you may verify that after the first step, the probability distribution of the particle on each node is simply

$$p_0 P = (\frac{1}{2}, 0, 0, \frac{1}{2})$$

and the probability distribution after 2 steps is $p_0 P^2$, ... The initial state $p_0$ and the transition matrix $P$ constitutes what is called a “Markov chain”. The study of its property boils down to study of matrix powering $P^n$ [2]. Note that by Jordan’s normal form, we can write $P = B^{-1}JB$. (Consult a linear algebra source if you need to look up Jordan’s normal form). Then, $P^n$ is simply $B^{-1}J^nB$. Therefore the convergence behavior of the Markov chain relies on the eigen-structure of the matrix $P$.

Here we have an important result called the Perron-Frobenius Theorem: If $P$ is the transition matrix of a (1)finite, (2)irreducible and (3)aperiodic Markov chain, then $P$ has an eigen-value of 1, and all other eigen-values has magnitude strictly less than 1.

The above is a simple form of the Perron-Frobenius Theorem. For the full-fledged theorem and its various proofs and implications, consult [M00]. For our purpose, the preceding theorem guarantees the convergence of the Markov chain.

Now, the other way around: an (impromptu) example to convert a Markov chain into a (directed) graph. ..

Note that equivalence between terms in two contexts:

[2] however, more recent developments on Markov chains rely on other intuitive, more “probabilistic” techniques such as coupling.

| Markov Chain | Graph (Random Walk) |
|---|---|
| Aperiodic | Non-bipartite |
| Irreducible | Strongly Connected |

Table 1: Equivalent terms in two contexts

# References


[GS82] GRIMMETT and STIRZAKER, “Probability and Random Processes”, Oxford Press, 1982

[M00] C.R. MACCLUER, “The Many Proofs and Applications of Perron’s Theorem”, SIAM Review, Vol. 42, No. 3, pp. 487-498, 2000