Stat210B: Theoretical Statistics	**Lecture Date: April 10, 2007**

Signed Rank Test and Likelihood Ratio

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1 Sign Rank Statistics

Let $T_n = \frac{1}{n}\sum_i a_{n,R^+_{n,i}} sign(X_i)$, e.g., $a_{n,i} = \mathbb{E}[\phi(U_{n(i)})], U_{n(i)}$ are ordered statistics of $Unif(0,1)$. Let $F^+(x) = 2F(x) - 1$ be the c.d.f. of $|X_i|$. Recall we assume the distribution is symmetric. Then,

$$T_n = \frac{1}{\sqrt{n}} \sum_i \phi(F^+(|X_i|)) sign(X_i) + o_{P_0}(1).$$

Choose ϕ in a smart way, note that $\frac{f'}{f}(|x|)sign(x) = \frac{f'}{f}(x)$, set

$$\phi(U) = -\frac{1}{\sqrt{I_f}} \frac{f'}{f}((F^+)^{-1}(U)),$$

then,

$$T_n = -\frac{1}{\sqrt{n}} \sum_i \frac{1}{I_f} \frac{f'}{f}(|X_i|) sign(X_i) + o_{P_0}(1) = -\frac{1}{\sqrt{n}} \frac{1}{I_f} \sum_i \frac{f'}{f}(X_i) + o_{P_0}(1),$$

which satisfies Theorem 2 in lecture 21, Apr 5, (Thm 15.4, p. 221 in van der Vaart (1998)). Thus, the signed rank test is asymptotically optimal.

Example 1. Laplace $f(x) \propto e^{-|x|}$. This implies $T_n = \frac{1}{\sqrt{n}} \sum_i sign(X_i)$, which is a sign test.

2 Likelihood Ratio Test

$$\widetilde{\Lambda}_n = 2 \log \frac{\sup_{\theta\in\Theta_1} \prod_i P_\theta(X_i)}{\sup_{\theta\in\Theta_0} \prod_i P_\theta(X_i)},$$

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where $\Theta = \Theta_0 \bigcup \Theta_1$. So Λ_n clips to 0, which does not matter because we reject for large Λ_n.

LAN approach

- introduce local parameter spaces.

$$\theta_n = \sqrt{n}(\theta + \frac{h}{\sqrt{n}}),$$

$$H_n = \sqrt{n}(\Theta - \eta) \quad \text{and} \quad H_{n,0} = \sqrt{n}(\Theta_0 - \eta).$$

- Write Λ_n using H_n and $H_{n,0}$.

$$\Lambda_n = 2 \sup_{h \in H_n} \log \frac{\prod_i P_{\eta+h/\sqrt{n}}(X_i)}{\prod_i P_\eta(X_i)} - 2 \sup_{h \in H_{n,0}} \log \frac{\prod_i P_{\eta+h/\sqrt{n}}(X_i)}{\prod_i P_\eta(X_i)}.$$

- LAN result give asymptotic expansions for the logs. (e.g., under qmd).
- Under suitable notion of convergence of sets, we expect Λ_n to converge to

$$\Lambda = \inf_{h \in H_0} (X-h)^T I_\eta (X-h) - \inf_{h \in H} (X-h)^T I_\eta (X-h),$$

  in which $X \sim N(h, I_\eta^{-1})$.
- The distribution of Λ_n under η corresponds to the distribution of Λ under $h = 0$.
- Under $h = 0$, $I_\eta^{1/2} X \sim N(0, I)$.

Lemma 2. *Lemma 16.6 in van der Vaart (1998). If $X \sim N_k(0, I)$, H_0 is a l-dim linear subspace of $\mathbb{R}^k$. Then*

$$\|X - H_0\|^2 \sim \chi^2_{k-l}.$$

- If η is an interior point of Θ, then H is all of $\mathbb{R}^k$, and second term in Λ vanishes.
- If $\sqrt{n}(\Theta_0 - \eta)$ converges to a linear subspace H_0 of dimension l, the asymptotic null distribution of Λ_n is χ^2_{k-l}.

Theorem 3. *Theorem 16.7 in van der Vaart (1998). suppose:*

1. *$\{P_\theta\}$ is qmd.*
2. *$|\log P_{\theta_1}(x) - \log P_{\theta_0}(x)| \leq \dot{l}(x)\|\theta_1 - \theta_2\|$, (makes LAN work) for some $\dot{l}$ such that $P_\eta \dot{l}^2 < \infty$ and $\theta_1, \theta_2 \in$ neiborhood of η. (makes EPT work.)*
3. *MLE's $\hat{\theta}_{n,0}$ and $\hat{\theta}_n$ are consistent.*
4. *$H_{n,0}$ and H_n converge to sets H_0 and H.*

Then

$$\Lambda \xrightarrow{\eta+h/\sqrt{n}} \Lambda \text{ for } X \sim N(h, I_\eta^{-1}).$$

What about power?

References

van der Vaart, A. W. (1998). *Asymptotic Statistics.* Cambridge University Press, Cambridge.