Stat 210B Homework Assignment 1 (due February 7)

1. Given densities p_n and q_n with respect to some measure μ, let X be distributed according to the distribution with density p_n. Define the *likelihood ratio* $L_n(X)$ as $L_n(X) = q_n(X)/p_n(X)$ for $p_n(X) > 0$, $L_n(X) = 1$ if $p_n(X) = q_n(X) = 0$ and $L_n(X) = \infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence.

2. Let $X_1, \ldots, X_n$ be a sample of size n from Beta$(\theta, 1)$, where $\theta > 0$. Let $\bar{X}_n$ denote the sample mean. The method-of-moments estimate of θ is $\hat{\theta}_n = \bar{X}_n/(1 - \bar{X}_n)$. Find its asymptotic distribution.

3. Suppose X_n is uniformly distributed on the set of points $\{1/n, 2/n, \ldots, 1\}$. Show that $X_n \xrightarrow{d} X$, where X is Un$(0, 1)$, where Un denotes the uniform distribution. Does $X_n \xrightarrow{P} X$?

4. Suppose that $F_n(x) \to F(x)$ for all x, F is continuous and strictly increasing so that $F^{-1}(\alpha)$ is unique for all $0 < \alpha < 1$. Show that

$$\sup\{|F_n^{-1}(\alpha) - F^{-1}(\alpha)| : \epsilon \leq \alpha \leq 1 - \epsilon\} \to 0$$

   for all $\epsilon > 0$. Here $F_n^{-1}(\alpha) = \inf\{x : F_n(x) \geq \alpha\}$.

5. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be an i.i.d. sample of 2-D random vectors (X, Y) where $0 < EX^4 < \infty$, $0 < EY^4 < \infty$. Let $\rho^2 = \text{cov}^2(X, Y)/\sigma_1^2\sigma_2^2$, where $\sigma_1^2 = \text{var}(X)$, $\sigma_2^2 = \text{var}(Y)$; and let $r^2 = \hat{C}^2/\hat{\sigma}_X^2\hat{\sigma}_Y^2$ where

$$\hat{C} = n^{-1}\sum(X_i - \bar{X})(Y_i - \bar{Y}), \quad \hat{\sigma}_X^2 = n^{-1}\sum(X_i - \bar{X})^2, \quad \hat{\sigma}_Y^2 = n^{-1}\sum(Y_i - \bar{Y})^2$$

   show that

   (a) If $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then $\sqrt{n}(r^2 - \rho^2) \xrightarrow{d} N(0, 4\rho^2(1 - \rho^2)^2)$ and, if $\rho \neq 0$, then $\sqrt{n}(r - \rho) \to N(0, (1 - \rho^2)^2)$.

   (b) If $\rho = 0$, $\sqrt{n}(r - \rho) \xrightarrow{d} N(0, 1)$.