CS 281A/Stat 241A Homework Assignment 1 (due September 11)

1. **Basic conditional independencies**

   Recall the definition of the set of *basic conditional independence statements* associated with a directed graphical model:

   $$\{X_i \perp\!\!\!\perp X_{\nu_i} \mid X_{\pi_i}\} \tag{1}$$

   where ν_i is the set of nodes that precede i in a topological ordering I, excluding the parents π_i.

   (a) Prove that ν_i contains all *ancestors* of i (excluding the parents π_i).

   (b) Prove that ν_i contains no *descendants* of i.

2. **Minimal conditioning**

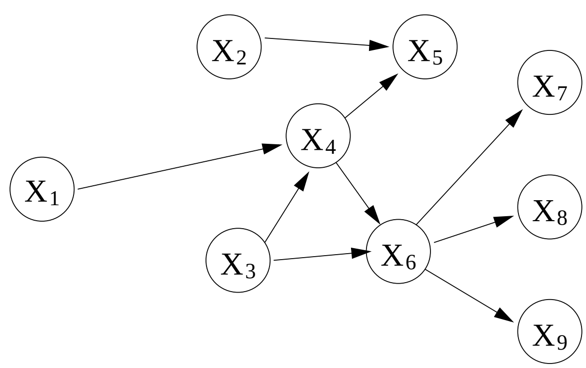
   For a given node X_i in a graphical model, what is the minimal set of nodes that renders the node conditionally independent of all of the other nodes in the graph? That is, what is the smallest set X_C such that $X_i \perp\!\!\!\perp X \backslash (X_C \cup X_i) \mid X_C$? (Note that X is the set of all nodes in the graph and that $X \backslash (X_C \cup X_i)$ is therefore the set of all nodes in the graph excluding X_i and X_C).

   (a) Do the problem for an undirected graph.

   (b) Do the problem for a directed graph.

3. **Moralization and elimination**

   Consider the directed graphical model below.



   (a) Moralize the graph, and show the resulting undirected graphical model.

   (b) Invoke UNDIRECTEDGRAPHELIMINATE on your moral graph, using the elimination ordering (7, 8, 9, 6, 3, 5, 4, 2, 1), and show the resulting reconstituted graph (the graph that includes all the edges added during the elimination process). You do not have to show the intermediate steps of the algorithm.

(c) Repeat (b), but using the elimination ordering (7, 6, 8, 9, 4, 3, 2, 5, 1).

(d) Which of the two elimination orderings, if used in ELIMINATE to calculate $p(x_1|x_7)$, do you think would result in a more time-efficient calculation? Which would result in a more space-efficient calculation? Briefly explain why.

4. **Tree representations**

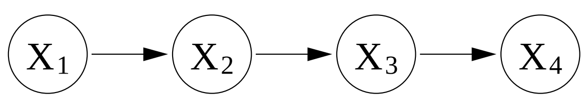
Consider an undirected tree. We know that it is not possible in general to parameterize a distribution on a tree using marginal probabilities as the potentials. It is, however, possible to parameterize such a distribution using ratios of marginal probabilities. In particular, let:

$$\begin{aligned}\psi(x_i) &= p(x_i) \\ \psi(x_i, x_j) &= \frac{p(x_i, x_j)}{p(x_i)p(x_j)},\end{aligned}$$

where i and j are neighbors in the tree, where $p(x_i)$ and $p(x_i, x_j)$ are a consistent set of marginal probabilities. Show that this setting of potentials yields a parameterization of a joint probability distribution on the tree under which $p(x_i)$ and $p(x_i, x_j)$ are marginals. What is Z under this parameterization?

5. **Elimination**

In this problem we'll work through the mechanics of using ELIMINATE for inference in a directed graphical model. Consider the directed graphical model below over the binary variables X_1, X_2, X_3 and X_4.



Let $p(X_1 = 0) = .5$ and $p(X_1 = 1) = .5$. For the local conditional probabilities, $p(x_{i+1} \mid x_i)$, use the following matrix:

	$X_i = 0$	$X_i = 1$
$X_{i+1} = 0$	.6	.2
$X_{i+1} = 1$	.4	.8

We will use ELIMINATE to compute $p(X_1 = 0|X_4 = 1)$.

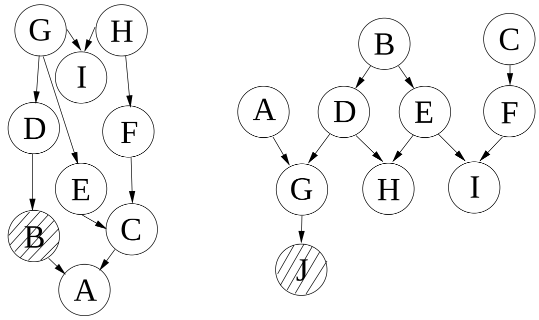


Figure 1: Bayes ball examples.

(a) Write out the summations involved in carrying out the elimination algorithm to find $p(x_1, x_4)$ and $p(x_4)$, as in Equations (3.8) to (3.15) in Chapter 3 of the text. Use the elimination ordering (4, 3, 2, 1).

(b) Using ELIMINATE (with $\bar{x}_4 = 1$), write down (as tables) the resulting intermediate terms, and $p(x_1, \bar{x}_4)$.

(c) Compute $p(X_1 = 0|X_4 = 1)$.

(d) You can alternatively do parts (a), (b) and (c) using Matlab or Splus.

6. ***d*-Connectivity.**

For the two graphs in Figure 1, determine which variables can be reached using the Bayes ball algorithm if we start at node A.