CS 281A/Stat 241A Homework Assignment 1 (due September 13)

1. **Induced graphs under conditioning and marginalization**

   (a) Consider a directed graph $G = (V, E)$. Consider a subset $A \subseteq V$. What is the induced graph structure corresponds to the conditional distribution $p(x_{V \setminus A} \mid x_A)$?

   (b) For a subset $A \subseteq V$, what induced graph structure corresponds to the marginal distribution $p(x_{V \setminus A})$?

   (c) Redo the problem for undirected graphs.

2. **Ising model**

   Consider an $n \times n$ lattice (a two-dimensional grid in which each interior node is linked to its four nearest neighbors). What is the best elimination order for such a lattice? What is the running time of the elimination algorithm for such a best order?

3. **Minimal conditional independence**

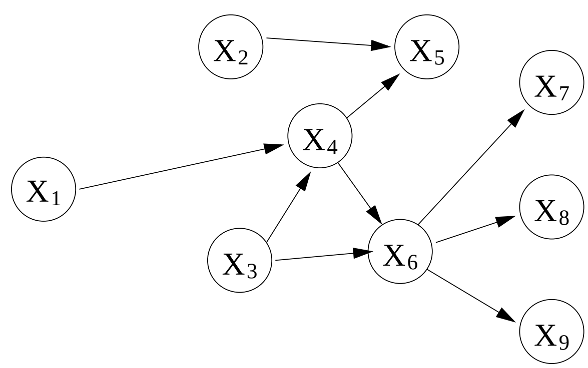
   For a given variable X_i in a graphical model, what is the minimal set of nodes that renders X_i conditionally independent of all of the other variables? That is, what is the smallest set C such that $X_i \perp\!\!\!\perp X_{V \setminus \{i\} \cup C} \mid X_C$? (Note that V is the set of all nodes in the graph, so that $V \setminus \{i\} \cup C$ is the set of all nodes in the graph excluding i and C).

   (a) Do the problem for an undirected graph.

   (b) Do the problem for a directed graph.

4. **Moralization and elimination**

   Consider the directed graphical model below.



   (a) Moralize the graph, and show the resulting undirected graphical model.

   (b) Invoke UndirectedGraphEliminate on your moral graph, using the elimination ordering (7, 8, 9, 6, 3, 5, 4, 2, 1), and show the resulting reconstituted graph (the graph that includes all the edges added during the elimination process). You do not have to show the intermediate steps of the algorithm.

   (c) Repeat (b), but using the elimination ordering (7, 6, 8, 9, 4, 3, 2, 5, 1).

   (d) Which of the two elimination orderings, if used in Eliminate to calculate $p(x_1|x_7)$, do you think would result in a more time-efficient calculation? Which would result in a more space-efficient calculation? Briefly explain why.

5. **Tree representations**

Consider an undirected tree. We know that it is not possible in general to parameterize a distribution on a tree using marginal probabilities as the potentials. It is, however, possible to parameterize such a distribution using ratios of marginal probabilities. In particular, let:

$$\begin{aligned} \psi(x_i) &= p(x_i) \\ \psi(x_i, x_j) &= \frac{p(x_i, x_j)}{p(x_i)p(x_j)}, \end{aligned}$$

where i and j are neighbors in the tree, where $p(x_i)$ and $p(x_i, x_j)$ are a consistent set of marginal probabilities. Show that this setting of potentials yields a parameterization of a joint probability distribution on the tree under which $p(x_i)$ and $p(x_i, x_j)$ are marginals. What is Z under this parameterization?

6. **Elimination**

In this problem we'll work through the mechanics of using ELIMINATE for inference in a directed graphical model. Consider the directed graphical model below over the binary variables X_1, X_2, X_3 and X_4.

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow X_4$$

Let $p(X_1 = 0) = .5$ and $p(X_1 = 1) = .5$. For the local conditional probabilities, $p(x_{i+1} \mid x_i)$, use the following matrix:

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{x_i} \\ & & 0 & 1 \\ \cline{3-4} x_{i+1} & 0 & .6 & .2 \\ \cline{3-4} & 1 & .4 & .8 \\ \cline{3-4} \end{array}$$

We will use ELIMINATE to compute $p(X_1 = 0|X_4 = 1)$, using the elimination ordering (4, 3, 2, 1). Implement ELIMINATE using R or Matlab. (Your implementation need not be entirely general; if you wish, you can restrict it to chains that are eliminated in backwards order).

(a) Compute $p(X_1 = 0|X_4 = 1)$.