CS 4: Lecture 21
Monday, April 10, 2006

The Project: Rocket Motion

The Lunar Lander project asks you to simulate the motion of a rocket. One way that rocket motion differs from pool balls is that the rocket propels itself by expelling fuel, so its mass decreases as it moves.

Let q(t) be the flow rate of the expelled gas, measured in kg/s, as a function of time. The flow rate is controlled by the rocket's driver.

Let m(t) be the mass of the rocket as a function of time. The mass can be determined by integrating the differential equation

$$\frac{dm(t)}{dt} = -q(t). \qquad (1)$$

Conservation of momentum is the principle that underlies the rocket's propulsion. The change in momentum of the fuel expelled from the rocket's tail is matched by increased momentum in the rocket.

If q(t) is constant over the duration of a timestep (which is always true in the Lunar Lander project), then Euler's method is all you need for time integration of this differential equation with perfect accuracy.

Let F_{re} be the force applied to the rocket by the exhaust, and F_{er} be the force applied to the exhaust by the rocket. By Newton's Third Law,

$$F_{re} = -F_{er}.$$

If the rocket experiences gravity, the total force on the rocket is

$$F_r(t) = -F_{er}(t) + g\,m(t),$$

where g is the gravitational acceleration, which points toward the planet or moon that exerts the gravitational force. (Note that g and the F's are vectors, and may point in different directions.)

The force on the exhaust is its flow rate times its velocity,

$$F_{er}(t) = q(t)\,v_e(t).$$

What is its velocity? Let w be the velocity of the exhaust, _relative_ to the rocket. The magnitude of w depends on chemical processes, and is typically roughly constant. Let v(t) be the velocity of the rocket as a function of time. The _absolute_ velocity of the exhaust is $v_e(t) = w + v(t)$, because the fuel was moving with the rocket until the moment it was exhausted.

In the Lunar Lander project, w is fixed at 1,700 m/s. The direction of w depends on how the rocket is oriented--the exhaust shoots out the rocket's tail. In Lunar Lander, the orientation of the lander is not always the same as the direction it's moving.

So we have

$$F_r(t) = -q(t)\,[w + v(t)] + m(t)\,g.$$

Normally, the next step would be to use Newton's Second Law, F = ma. But there's a surprising problem...

F = ma is WRONG when m changes as a function of time!

So what can we do instead? Well, force is the rate of change of momentum. Let p(t) be the rocket's momentum as a function of time. Then

$$\frac{dp(t)}{dt} = -q(t)\,[w + v(t)] + m(t)\,g.$$

Momentum is mass times velocity, i.e., p(t) = m(t) v(t). By the chain rule,

$$\frac{dp(t)}{dt} = m(t)\frac{dv(t)}{dt} + \frac{dm(t)}{dt}v(t) = -q(t)\,[w + v(t)] + m(t)\,g.$$

Notice that one of the derivatives is the acceleration a(t) = dv/dt. So although F = ma is wrong, the truth is something close: F = ma + m'v. That's how Newton's Second Law changes when mass is not constant.

Recall from (1) that dm/dt = - q(t). Therefore, two of the terms cancel each other out, and we have

$$m(t)\frac{dv(t)}{dt} = -q(t)\,w + m(t)\,g.$$

Dividing both sides by m(t) gives

$$\frac{dv(t)}{dt} = -\frac{q(t)}{m(t)}\,w + g. \qquad (2)$$

We can integrate this differential equation with Euler's Method and solve for the rocket's velocity. However, because the mass changes value during each timestep, we'll get a more accurate result if we use Heun's method.

Knowing the velocity, we can also advance the position x(t) of the rocket.

$$\frac{dx(t)}{dt} = v(t).$$

Again, Euler's Method is a good start, but Heun's Method is more accurate.

Heun's Method

Heun's Method gives better accuracy than Euler's Method for computing v(t) and x(t). Here's how you use it. First, compute m(t + T) (after one timestep) with Euler's Method. This gives an accurate result because q(t) doesn't vary during a timestep.

$$m(t + T) = m(t) - q(t)\ T.$$

Then use the trapezoid rule to integrate (2) and compute an accurate v(t + T).

$$v(t + T) \sim v(t) - T\ q(t)\ \left[\frac{1}{2\ m(t)} + \frac{1}{2\ m(t + T)} \right] w + T\ g.$$

Finally, use the trapezoid rule to compute an accurate x(t + T).

$$x(t + T) \sim x(t) + \frac{T}{2} [\ v(t) + v(t + T)\ ].$$

Note that because v doesn't depend on x, and m doesn't depend on v or x, we don't have to use an Euler "guess" (recall Lecture 9) to use Heun's method here.