CS 61B Lab 9
March 19-21, 2012

Goal: to practice proving asymptotic (big-Oh) results. The class notes from Lecture 20 may come in handy; however, we will try to be even more rigorous here than in lecture.

Recall the definition of big-Oh:

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|=============================================================================|
| O(f(n)) is the SET of ALL functions T(n) that satisfy:                      |
|                                                                             |
|   There exist positive constants c and N such that, for all n >= N,         |
|                              T(n) <= c f(n)                                 |
|=============================================================================|
```

EXAMPLE

Formally prove that n^2 + n + 1 is in O(n^2).

Solution:
Let T(n) = n^2 + n + 1. Let f(n) = n^2.

Choose c = 3, and N = 1. Then, we know T(n) is in O(n^2) if we can prove

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                               T(n) <= c f(n),
    or equivalently,    n^2 + n + 1 <= 3 n^2,                for all n >= 1.
```

Is this inequality true? Well, for any n >= 1, we know that 1 <= n <= n^2. Hence, all of the following are true:

```
                                  1 <= n^2
                                  n <= n^2
                                n^2  = n^2
```

Adding the left and right sides of these inequalities together, we have

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                        n^2 + n + 1 <= 3 n^2, which completes the proof.
```

Part I: (1 point)

Formally prove that 2^n + 1 is in O(4^n - 16).

HINT: You may assert without proof that, for all n >=1, 2^n >= 1. (You may also assert without proof that 4^n and 2^n are monotonically increasing, if you find it useful.)

Part II: (1 point)

Formally prove that if f(n) is in O(g(n)), and g(n) is in O(h(n)), then f(n) is in O(h(n)).

NOTE: The values of c and N used to prove that f(n) is in O(g(n)) are not necessarily the same as the values used to prove that g(n) is in O(h(n)). Hence, assume that there are positive c', N', c'', and N'' such that

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    f(n) <= c' g(n)        for all n >= N', and
    g(n) <= c'' h(n)       for all n >= N''.
```

Part III: (2 points)

Formally prove that 0.01 n^2 - 1 is NOT in O(n).

We need to show that, no matter how large we choose c and N, we will never obtain the desired inequality. We cannot prove this by picking a specific value of c and N. Instead, we must study how the two functions behave as n approaches infinity.

Let T(n) = 0.01 n^2 - 1, and let f(n) = n. Prove that

```
                    c f(n)
            lim     ------ = 0,
        n->infinity  T(n)
```

no matter how large we choose c to be. You will need to scale both the numerator and the denominator by a well-chosen multiplier to get the result.

Use this result to show that there are no values c, N such that T(n) <= c f(n) for all n >= N.

Postscript

The functions |cos(n)| and |sin(n)| are interesting, because neither is dominated by the other. Can you informally suggest why |cos(n)| is not in O(|sin(n)|), and |sin(n)| is not in O(|cos(n)|)?

How would you prove that, for all n >=1, 2^n >= 1? (Hint: use calculus.)