Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **View of a polygon.** Let P be a simple polygon and x a point outside P in the plane. An edge *faces* x if x lies outside its face figure, which is the closed half-plane bounded by the line of the edge and locally on the same side as the polygon. Consider maximal sequences of edges facing x. In Figure 1 these sequences begin and end at marked vertices and they are labeled either -1, 0, or $+1$. The endpoint of a sequence is marked with a hollow vertex if the sequence turns away from x at that vertex, and with a solid vertex if the sequence turns towards x at that vertex. The *label* of a sequence

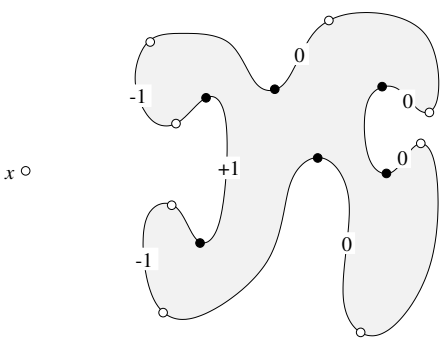

Figure 1: A stylized polygon.

is -1 plus the number of solid endpoints.

   (i) Prove that the sum of labels is -1 for every simple polygon and every point x outside that polygon.

   (ii) What is the sum of labels if the polygon has β_0 components and β_1 holes?

2. **Alternating angle sums.** Let $abcd$ be a tetrahedron and consider the solid angles at the four vertices and the dihedral angles at the six edges. Normalize so all angles lie between 0 and 1.

   (i) Prove that the sum of solid angles minus the sum of dihedral angles is -1.

   (ii) Show that the sum of solid angles can be anywhere between 0 and $\frac{1}{2}$, but nowhere else.

   (iii) Show that the sum of dihedral angles can be anywhere between 1 and $\frac{3}{2}$, but nowhere else.

3. **Spherical tetrahedra.** A *spherical tetrahedron* is the intersection of a 3-sphere with a tetrahedral cone whose apex lies at the center of the 3-sphere in $\mathbb{R}^4$. The solid and dihedral angles of the spherical tetrahedron are the corresponding angles between pairs of facets and triplets of facets of the cone. We assume all angles are normalized between 0 and 1. Show that the sum of solid angles minus the sum of dihedral angles is -1, just as for ordinary tetrahedra in $\mathbb{R}^3$.

4. **Independent complex.** Let S be a collection of sets. A subcollection $T \subseteq S$ is *independent* if for every $U \subseteq T$ there is an element in all sets of U that is in no set of $T - U$:

$$\bigcap U - \bigcup (T - U) \quad \neq \quad \emptyset.$$

   Let A be the system of independent collections $T \subseteq S$. Show that A is an abstract simplicial complex.

5. **Independent collections.** Suppose the collection S in the previous exercise contains geometric sets of a particular kind. What is the maximum size of an independent subcollection is these sets are

   (i) intervals in $\mathbb{R}$,

   (ii) axes-aligned rectangles in $\mathbb{R}^2$,

   (iii) triangles in $\mathbb{R}^2$,

   (iv) convex sets in $\mathbb{R}^2$?