Meeting 2 January 21, 1999

Delaunay Triangulations

`Topics:` Voronoi diagram, Delaunay triangulation, degeneracy, circles and power, acyclicity.

Voronoi diagram. Given a finite set of points in the plane, the idea is to assign to each point a region of influence in such a way that the regions decompose the plane. To describe a specific way to do that let $S \subseteq \mathbb{R}^2$ be a set of n points and define the *Voronoi region* of $p \in S$ as the set of points $x \in \mathbb{R}^2$ that are at least as close to p as to any other point in S:

$$V_p \quad = \quad \{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\|, \ \forall q \in S\},$$

see Figure 1. Consider the half-plane of points at least

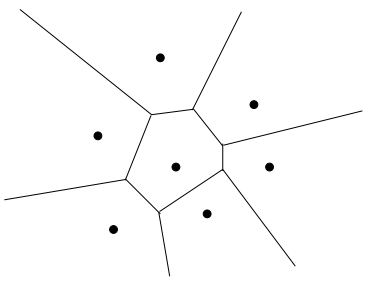

Figure 1: Seven points which define the same number of Voronoi regions. One of the regions is bounded because the defining point is surrounded by others.

as close to p as to q: $H_{pq} = \{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\|\}$. The Voronoi region of p is the intersection of half-planes H_{pq}, for all $q \in S - \{p\}$. It follows that V_p is a convex polygonal region, possibly unbounded, with at most $n - 1$ edges.

Each point $x \in \mathbb{R}^2$ has at least one nearest point in S, so it lies in at least one Voronoi regions. It follows that the Voronoi regions cover the entire plane. Two Voronoi regions lie on opposite sides of the perpendicular bisector separating the two generating points. It follows that Voronoi regions do not share interior points, and if a point x belongs to two Voronoi regions then it lies on the bisector of the two generators.

Delaunay triangulation. We get a dual diagram if we draw a straight *Delaunay edge* connecting points $p, q \in S$ iff their Voronoi regions intersect along a common line segment, see Figure 2. In general, the Delaunay edges decompose the convex hull of S into triangular regions, which are referred to as *Delaunay triangles*.

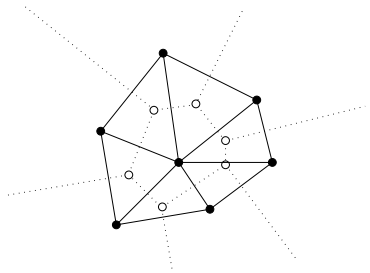

Figure 2: The Voronoi edges are dotted and the dual Delaunay edges are solid.

To count the Delaunay edges we use some results on *planar graphs* defined by the property that their edges can be drawn without crossings in the plane. It is true that no two Delaunay edges cross each other, but to avoid an argument we drawing each Delaunay edge from one endpoint straight to the midpoint of the shared Voronoi edge and then straight to the other endpoint. Now it is trivial that not two of these edges cross. Using Euler's relation it can be shown that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges and at most $2n - 4$ faces. The same bounds hold for the number of Delaunay edges and triangles. There is a bijection between the Voronoi edges and the Delaunay edges, so $3n - 6$ is also an upper bound on the number of Voronoi edges. Similarly, $2n - 4$ is an upper bound on the number of Voronoi vertices.

Degeneracy. There is an ambiguity in the definition of Delaunay triangulation if four or more Voronoi re-

gions meet at a common vertex u, see Figure 3. The

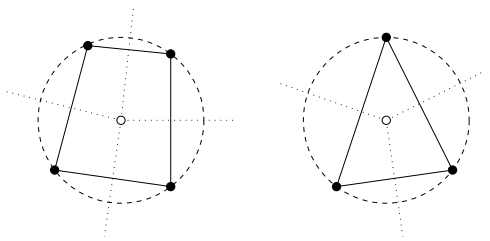

Figure 3: To the left four dotted Voronoi edges meet at a common vertex and the dual Delaunay edges bound a quadrilateral. To the right we have the general case where only three Voronoi edges meet at a common vertex and the Delaunay edges bound a triangle.

points generating the four or more regions all have the same distance from u; they lie on a common circle around u. Probabilistically, the chance of picking even just four points on a circle is 0 because the circle defined by the first three points has zero measure in $\mathbb{R}^2$. A common way to say the same thing is that four points on a common circle form a *degeneracy* or a *special case*. An arbitrarily small perturbation suffices to remove the degeneracy and to reduce the special to the general case.

We will often assume *general position*, which is the absence of any degeneracy. This really means that we delay the treatment of degenerate cases to later. The treatment is eventually done with a perturbation, which can be actual or conceptual, or by exhaustive case analysis.

Circles and power. For now we assume general position. For a Delaunay triangle abc consider the circumcircle, which is the unique circle passing through a, b, and c. Its center is the corresponding Voronoi vertex, $u = V_a \cap V_b \cap V_c$, and its radius is $\varrho = \|u-a\| = \|u-b\| = \|u-c\|$, see Figure 3. We call the circle *empty* because it encloses no point of S. It turns out that empty circles characterize Delaunay triangles.

CLAIM 1. Let $S \subseteq \mathbb{R}^2$ be finite and in general position, and let $a, b, c \in S$ be three points. Then abc is a Delaunay triangle iff the circumcircle of abc is empty.

It is not entirely straightforward to see that this is true, at least not at the moment. Instead of proving Claim 1 we focus our attention on a new concept of distance from a circle. The *power* of a point $x \in \mathbb{R}^2$ from a circle U with center u and radius ϱ is

$$\pi_U(x) \quad = \quad \|x-u\|^2 - \varrho^2.$$

If x lies outside the circle then $\pi_U(x)$ is the square length of a tangent line segment connecting x with U. In any case, the power is positive outside the circle, zero on the circle, and negative inside the circle. We sometimes think of a circle as a weighted point and of the power as a weighted distance to that point. Given two circles, the set of points with equal power from both is a line. Figure 4 illustrates three different arrangements of two circles with bisector.

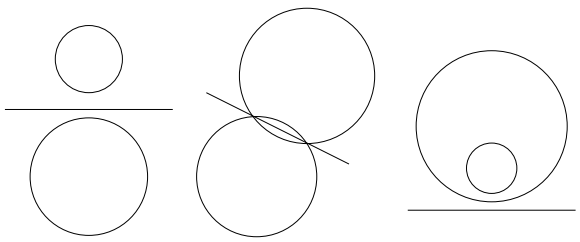

Figure 4: Three times two circles with bisector. From left to right: two disjoint and non-nested circles, two intersecting circles, two nested circles.

Acyclicity. We use the notion of power to prove an acyclicity result for Delaunay triangles. Let $x \in \mathbb{R}^2$ be an arbitrary but fixed viewpoint. We say a triangle abc *lies in front of* another triangle def if there is a half-line starting at x that first passes through abc and then through def, see Figure 6. We write $abc \prec def$ if abc lies in front of def. The set of Delaunay triangles together with $\prec$ forms a relation. General relations have cycles, which are sequences $\tau_0 \prec \tau_1 \prec \ldots \prec \tau_k \prec \tau_0$. Such cycles can also occur in general triangulations, as

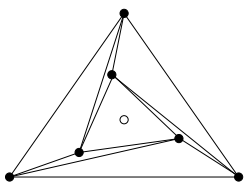

Figure 5: The hollow circle in the middle is the viewpoint for which the three skinny triangles form a cycle in the in front relation.

illustrated in Figure 5, but they cannot occur if the triangles are defined by empty circumcircles.

LEMMA 2. The in-front relation for the set of Delaunay triangles define by a finite set $S \subseteq \mathbb{R}^2$ is acyclic.

PROOF. We show that $abc \prec def$ implies that the power of x from the circumcircle of abc is less than the power

from the circumcircle of def. Define $abc = \tau_0$ and write $\pi_0(x)$ for the power of x from the circumcircle of abc. Similarly define $def = \tau_k$ and $\pi_k(x)$. Because S is finite,

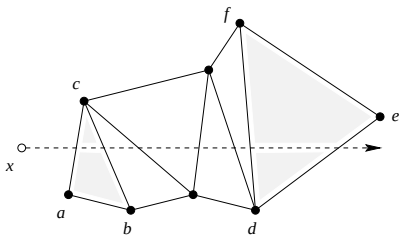

Figure 6: Triangle abc lies in front of triangle def. If abc and def belong to a Delaunay triangulation then there is a sequence of triangles between them that all intersect the half-line.

we can choose a half-line that starts at x, passes through abc and def, and contains no point of S. It intersects a sequence of Delaunay triangles:

$$abc = \tau_0 \prec \tau_1 \prec \ldots \prec \tau_k = def.$$

For any two consecutive triangles the line of equal power to the two circumcircles contains the common edge. Because the third point of τ_{i+1} lies outside the circumcircle of τ_i we have $\pi_i(x) < \pi_{i+1}(x)$, for $0 \leq i \leq k-1$. Hence $\pi_0(x) < \pi_k(x)$. The acyclicity of the relation follows because real numbers cannot increase along a cycle. □

Bibliographic notes. Voronoi diagrams are named after the Russian mathematician Georges Voronoi, who published two seminal papers at the beginning of this century [5]. However, the same concept was discussed about half a century earlier by P. G. L. Dirichlet, and there are unpublished notes by René Descartes suggesting he already used Voronoi diagram in the first half of the seventeenth century. Delaunay triangulations are named after the Russian mathematician Boris Delaunay (also Delone), who dedicated his paper on empty spheres to Georges Voronoi [2]. Article [1] by Franz Aurenhammer offers a nice survey of Voronoi diagrams and their algorithmic applications. The acyclicity of Delaunay triangulations in arbitrary dimension was first proved by Edelsbrunner [3]. This property has found applications in computer graphics. In particular the 3-dimensional case is exploited for the visualization of diffuse volume [4, 6].

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