Preserving Topology

`Topics:` manifolds, manifolds with boundary, open books, boundary, 2-complexes.

Manifolds. Suppose K is a 2-complex that triangulates a 2-manifold. Then every point $x \in |K|$ has a neighborhood homeomorphic to an open disk. To avoid lengthy sentences we just say the neighborhood *is* an open disk. This implies that in particular the star of every vertex u is an open disk. Strictly speaking this statement makes sense only if we replace the star by its underlying space, which we define as the union of simplex interiors, which is the set difference between the underlying spaces of two complexes:

$$\begin{aligned} |\operatorname{St} u| &= \bigcup_{\tau \in \operatorname{St} u} \operatorname{int} \tau \\ &= |\operatorname{Cl} \operatorname{St} u| - |\operatorname{Cl} \operatorname{St} u - \operatorname{St} u|. \end{aligned}$$

As it turns out the condition on vertex stars is sufficient to guaranteed that $|K|$ is a 2-manifold.

Claim 1. $|K|$ is a 2-manifold iff $|\operatorname{St} u| \approx \mathbb{R}^2$ for every vertex $u \in K$.

Now consider the contraction of an edge ab of K. Whether or not the contraction preserves the topological type depends on how the links of a and b meet. On a 2-manifold the link of each vertex is a circle. In Figure 4 to the left the two circles intersect in two points and the contraction preserves the topological type. To the right the circles intersect in a point and an edge, and in this case the contraction pinches the manifold along a newly formed edge which forms the base of a fin similar to the one in Figure 7. The condition that distinguishes topology preserving edge contractions from others is that the vertex links intersect in the link of the edge.

Theorem 2a. Let K be the triangulation of a 2-manifold. The contraction of $ab \in K$ preserves the topological type iff $\operatorname{Lk} a \cap \operatorname{Lk} b = \operatorname{Lk} ab$.

A proof of the sufficiency of the link condition will be given in the next lecture.

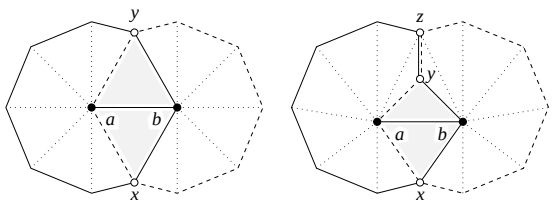

Figure 4: The edges of the link of a are solid and those of the link of b are dashed.

Manifolds with boundary. A triangulation K of a manifold with non-empty boundary also has vertices whose stars are open half-disks: $|\operatorname{St} u| \approx \mathbb{H}^2$. To keep the number of cases small we add a dummy vertex, ω, and the cone from ω to each boundary circle. This idea is illustrated in Figure 5. The boundary of $|K|$ consists

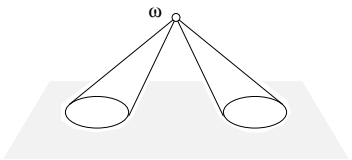

Figure 5: The two holes in the manifold are filled by adding the cone from ω to the circles bounding the holes.

of $\ell \geq 1$ circles triangulated by cycles $C_i \subseteq K$. We fill the holes by adding the cone from ω to every cycle:

$$K^\omega = K \cup (\omega \cdot \bigcup_{i=1}^{\ell} C_i).$$

In K^ω every vertex star is an open disk except possibly the star of ω. We denote the link of a vertex u in K^ω as $\operatorname{Lk}^\omega u$. The condition that distinguishes topology preserving edge contractions from others is now the same as for manifolds.

Theorem 2b. Let K be the triangulation of a 2-manifold with boundary. The contraction of $ab \in$

K preserves the topological type iff $\mathrm{Lk}^\omega a \cap \mathrm{Lk}^\omega b = \mathrm{Lk}^\omega ab$.

The proof of this result is only mildly more complicated than that of the weaker Theorem 2A.

Open books. To attack the problem for general 2-complexes we need a better understanding of the different types of neighborhoods that are possible. We classify stars using a new type of space. The *open book with p pages* is the topological space $\mathbb{K}^2_p$ homeomorphic to the union of p copies of $\mathbb{H}^2$ glued along the common boundary line. For example, the open book with one page is the open half-disk and the open book with two pages is the open disk. The *order* of a simplex $\tau \in K$ is

$$\mathrm{ord}\,\tau \quad = \quad \begin{cases} 0 & \text{if } |\,\mathrm{St}\,\tau\,| \approx \mathbb{R}^2, \\ 1 & \text{if } |\,\mathrm{St}\,\tau\,| \approx \mathbb{K}^2_p,\ p \neq 2, \\ 2 & \text{otherwise.} \end{cases}$$

Figure 6 illustrates the definition with sketches of four vertex stars. The order of an edge in a 2-complex can only be 0 or 1, and the order of a triangle is always 0.

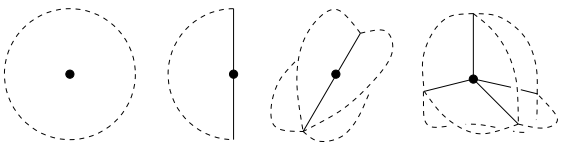

Figure 6: The underlying space of the vertex star in (a) is an open disk, in (b) is an open half-disk, in (c) is an open book with 4 pages, and in (d) is not an open book. The corresponding order of the vertex is 0 in (a), 1 in (b), 1 in (c), and 2 in (d).

Boundary. We generalize the notion of boundary in such a way that only triangulations of 2-manifolds have no boundary. At the same time we use the order information to distinguish between different types of boundaries. Specifically, the *j-th boundary* of a 2-complex K is the collection of all simplices with order j or higher:

$$\mathrm{Bd}_j K \quad = \quad \{\sigma \in K \mid \mathrm{ord}\,\sigma \geq j\}.$$

As an example consider the shark fin complex shown in Figure 7. It is constructed by gluing two closed disks along a simple path. This path is a contiguous piece of the boundary of one disk (the fin) and it lies in the interior of the other disk. Note that $\|K\|$ is a 2-manifold

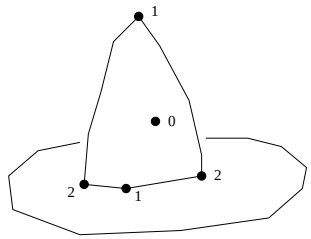

Figure 7: The shark fin 2-complex. A few of the vertices are high-lighted and marked with their order.

with boundary iff $\mathrm{Bd}_1 K = \mathrm{Bd}_2 K = \emptyset$. The 2-nd boundary of a 2-manifold with boundary is empty, but there are other spaces with this property. For example, the sphere together with its equator disk has empty 2-nd boundary. Its 1-st boundary is a circle of edges and vertices (the equator) whose stars are open books of 3 pages each.

2-complexes. We are now ready to study conditions under which an edge contraction in a general 2-complex preserves the topological type of that complex. As it turns out there does not exist a local condition that is sufficient and necessary, but there is a characterizing local condition for a more restrictive notion of type preservation. Let L be the 2-complex obtained from K by contracting an edge $ab \in K$. A *local unfolding* is a homeomorphism $f : \|K\| \to |L|$ that differs from the identity only outside the star of $\overline{ab}$, that is, $f(x) = x$ for all $x \in |K - \mathrm{St}\,\overline{ab}\,|$. The condition refers to links in $K^\omega = K \cup (\omega \cdot \mathrm{Bd}_1 K)$ and in $G^\omega = \mathrm{Bd}_1 K \cup (\omega \cdot \mathrm{Bd}_2 K)$. We denote the link of a simplex τ in K^ω by $\mathrm{Lk}^\omega_0 \tau$ and the link of τ in G^ω by $\mathrm{Lk}^\omega_1 \tau$.

THEOREM 2C. Let K be a 2-complex, ab an edge of K, and L the complex obtained by contracting ab. There is a local unfolding $|K| \to |L|$ iff

(i) $\mathrm{Lk}^\omega_0 a \cap \mathrm{Lk}^\omega_0 b = \mathrm{Lk}^\omega_0 ab$ and

(ii) $\mathrm{Lk}^\omega_1 a \cap \mathrm{Lk}^\omega_1 b = \emptyset$.

Instead of proving Theorem 2C, which is a bit tedious in any case, we show that there cannot be a similar condition that recognizes the existence of a general homeomorphism $|K| \to \|L\|$. The example we use is the folding chair complex displayed in Figure 8. Before the contraction of ab it consists of five triangles in the star of x and four disks U, V, Y, Z glued to the link of x. Vertices a and b belong to the 1-st boundary, but ab does not. It follows that ω violates condition (i) of Theorem 2C and there is therefore no local unfolding from $|K|$

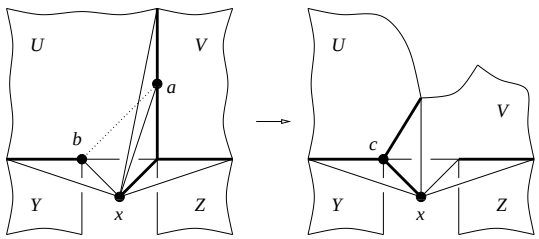

Figure 8: The folding chair complex. The bold edges belong to three triangles each.

to $|L|$. After the contraction there is one less triangle in the star of x, U loses two triangles, and V, Y, Z are unchanged. The contraction of ab exchanges left and right in the asymmetry of the complex. We can find a homeomorphism $\|K\| \to |L\|$ that acts like a mirror and maps U to V, V to U, Y to Z, Z to Y. The homeomorphism is necessarily global and to detect it we can force any algorithm to look at every triangle of K.

Bibliographic notes. The material of this lecture is taken from a recent paper by Dey et al. [1]. It studies edge contraction in general simplicial complexes and proves results for 2- and for 3-complexes. The order of a simplex has already been defined in 1960 by Whittlesey [4], although in different words and notation. He uses the concept to study the topological classification of 2-complexes. O'Dunlaing et al. [2] use his results to show that deciding whether or not two 2-complexes have the same topological type is just as hard as deciding whether or not two graphs are isomorphic. No polynomial time algorithm is known, but it is also not known whether the graph isomorphism problem is NP-complete [3].

[1] T. K. Dey, H. Edelsbrunner, S. Guha and D. V. Nekhayev. Topology preserving edge contraction. Report rgi-tech-98-018, Raindrop Geomagic, Research Triangle Park, North Carolina, 1998.

[2] C. O'Dunlaing, C. Watt and D. Wilkins. Homeomorphism of 2-complexes is equivalent to graph isomorphism. Rept. TCDMATH 98-04, Math. Dept., Trinity College, Ireland, 1998.

[3] M. R. Garey and D. S. Johnson. *Computers and Intractibility. A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, California, 1979.

[4] E. F. Whittlesey. Finite surfaces: a study of finite 2-complexes. *Math. Mag.* **34** (1960), 11–22 and 67–80.