Two-dimensional Alpha Shapes

`Topics:` circles and disks, Voronoi decomposition, growing disks, filter, computing connectivity.

Circles and disks. The reconstruction of curves from sampled points is one of several applications of alpha shapes. They define the shape of a finite set $S \subseteq \mathbb{R}^2$ using circles of radius $\alpha > 0$. We refer to these as α-*circles* and recall that a circle is said to be empty if no point of S lies inside. By definition, an edge pq belongs to the α-shape of S if there is an empty α-circle that passes through $p, q \in S$ in such a way that p and q cut the circle into a unique shorter and longer arc and there is no other point of S on the shorter arc. Figure 11 illustrates the definition. The edges form a graph that decompose the plane into regions, and there is a sense that each region is either foreground or background. It

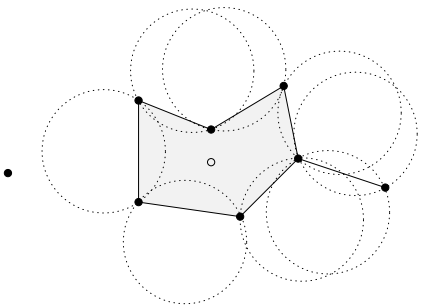

Figure 11: The α-shape of nine points. The dotted circles are certificates for the membership of the solid edges in the α-shape. The shaded region is foreground.

is foreground if there is not enough space to accommodate an empty α-circle, and it is background otherwise.

Instead of making this intuitive idea concrete we switch to a dual way of looking at alpha shapes. Note that an α-circle passes through points $p, q \in S$ iff its center lies on the intersection of the two α-circles centered at p and q. This observation motivates the definition of an α-*disk* as an open disk with radius $\alpha = 0$. Consider the union of α-disks centered at points of S as illustrated in Figure 12. The boundary of the union consists of circular arcs that meet at corner points. The

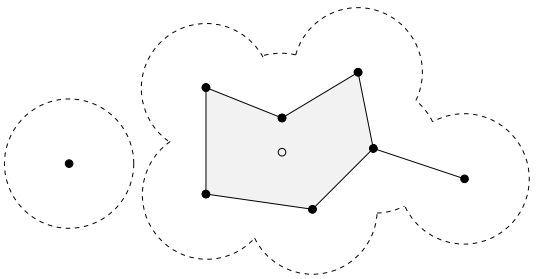

Figure 12: Union of α-disks and dual α-shape. The corners where the circular arcs meet are the centers of the certifying circles in Figure 11.

corners correspond to edges and the arcs correspond to vertices of the α-shape. The correspondence is not quite a bijection, but we can make it a bijection if we take duplicates of edges and vertices that are exposed from different directions.

Voronoi decomposition. We decompose the union of disks into convex regions using the Voronoi diagram. Let U_p be the α-disk with center p, recall that V_p is the Voronoi region of p, and define $U = \bigcup_{p \in S} U_p$.

Claim 1. $V_q \cap U = V_q \cap U_q$.

Proof. Let $x \in \mathbb{R}^2$ be a point in V_q. By definition of Voronoi region we have $\|x - q\| \leq \|x - p\|$ for all $p \in S$. Hence $x \in U$ only if $x \in U_q$. In other words,

$$x \in V_q \cap U \quad \Longrightarrow \quad x \in V_q \cap U_q.$$

The other direction is clear because $U_q \subseteq U$. ∎

Figure 13 illustrates the decomposition of the disk union into convex regions. The alpha shape can now be constructed by taking the boundary of the union of all Delaunay triangles, edges, vertices whose dual Voronoi cells have a non-empty intersection with the disk union.

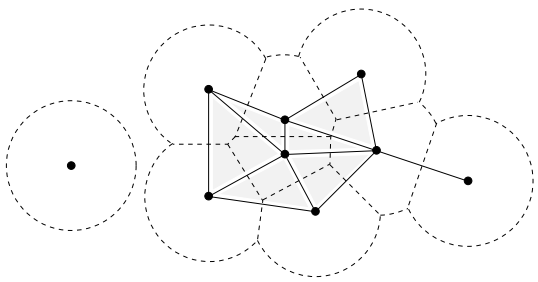

Figure 13: Voronoi decomposition of the disk union and the dual α-complex.

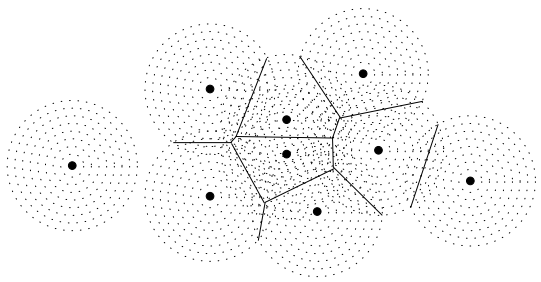

Figure 14: The corner points on the boundary of the disk union trace out the edges of the Voronoi diagram.

A succinct description of this construction can be given without reference to the Delaunay triangulation. Let the set of regions be $R = \{U_q \cap V_q \mid q \in S\}$ and assume general position so that no four regions in R have a non-empty intersection. The *α-complex* is the nerve of R geometrically realized by mapping $U_q \cap V_q$ to the generating point q:

$$K(\alpha) \quad = \quad \{\operatorname{conv} \varphi(T) \mid T \in \operatorname{Nrv} R\},$$

where $\varphi(U_q \cap V_q) = q$, see Figure 13. The *$\alpha$-shape* is the underlying space of $K(\alpha)$. The main advantage of this definition over the more intuitive earlier one is the fact that it generalizes to three and higher dimensions without any change.

Growing disks. The α-shape and the α-complex are defined for every positive radius α. If we increase α we grow the regions $U_q \cap V_q$, so also the nerve can only grow.

CLAIM 2. $K(\alpha_1) \subseteq K(\alpha_2)$ for all $0 < \alpha_1 \leq \alpha_2$.

The growth of the nerve happens in discrete steps, namely whenever the boundary of the disk union encounters a new Voronoi vertex, edge, or region. Figure 14 illustrates the growth process by drawing circles of various radii around the points. For infinite radius the disks cover the entire plane, the alpha complex is the Delaunay triangulation, and the alpha shape is the convex hull of S.

Filter. Let us equate the radius α with time and think of growth as an evolutionary process. For each Delaunay triangle, edge, vertex σ there is a time $t(\sigma)$ such that σ belongs to the α-complex iff $t(\sigma) < \alpha$. We can therefore sort the Delaunay simplices as $\sigma_1, \sigma_2, \ldots, \sigma_m$ such that $t(\sigma_i) < t(\sigma_j)$ implies $i < j$. Each alpha complex is then a prefix of this sequence.

A *filter* of a simplicial complex L is an ordering of the simplices as $\tau_1, \tau_2, \ldots, \tau_n$ such that $L_j = \{\tau_1, \tau_2, \ldots, \tau_j\}$ is a complex for every j. The sequence of prefixes, $L_0, L_1, \ldots, L_n$, is a *filtration* of L. For example an ordering where lower-dimensional simplices precede higher-dimensional ones is a filter. The sequence of α-complexes is almost a filtration of the Delaunay triangulation, except that contiguous α-complexes may differ in more than one simplex. This happens even if the points in S are in general position, and Figure 15 shows a small example. The edge qr and the triangle pqr enter

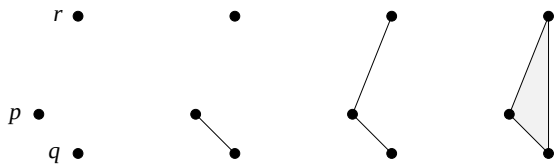

Figure 15: The four non-empty alpha complexes of three points forming an obtuse triangle.

the alpha complex at the same moment in time. That moment is the point in time when the boundary of the disk union sweeps over the circumcenter of pqr. The boundary encounters the Voronoi vertex dual to pqr at the same time as it encounters the Voronoi edge dual to qr that starts at that Voronoi vertex. To be sure that the ordering of Delaunay simplices results in a filter we sort them lexicographically, first by time stamp and if there is a tie by dimension. Remaining ties are broken arbitrarily. Let $K_0, K_1, \ldots, K_m$ be the resulting filtration, which contains the α-complexes as a scattered subsequence.

Computing connectivity. Let β_0 be the number of components and β_1 the number of holes of an alpha complex $K_j = K(\alpha)$. It is easy to compute both numbers using the filter of Delaunay simplices.

```
β0 = β1 = 0;
for i = 1 to j do k = dim σi;
  case k == 0: β0++;
  case k == 1:
    if σi belongs to a cycle in Ki then β1++
                                   else β0--
    endif;
  case k == 2: β1--
endfor.
```

If we run the algorithm for $j = m$ we get the values of β_0 and β_1 for *all* α-complexes. The running time depends on how fast we can determine whether or not an edge belongs to a cycle of the current complex. Using a union-find data structure representing the components as sets of vertices we can make such a determination in time at most proportional to the inverse of Ackermann's function, which for all practical purposes is constant. The running time of the algorithm is therefore at most proportional to m times the inverse of the Ackermann function for m, which is usually written as $O(m \cdot \alpha(m))$.

We note that β_0 and β_1 are known as the Betti numbers of K_j. Let s_k be the number of k-simplices in K_j. The Euler characteristic is

$$\begin{aligned} \chi &= s_0 - s_1 + s_2 \\ &= s_0 - (s_1' + s_1'') + s_2 \\ &= (s_0 - s_1') - (s_1'' - s_2) \\ &= \beta_0 - \beta_1, \end{aligned}$$

where s_1' is the number of edges that connect different components and s_1'' is the numer of edges that belong to a cycle the moment they are added to the complex. The algorithm generalizes to higher dimensions and leads to the celebrated Euler-Poincaré formula, which says that χ is the alternating sum of Betti numbers. At this moment we do not have the necessary background to discuss this formula in amount of detail.

Bibliographic notes. Alpha shapes in two dimensions have been introduced in 1983 by Edelsbrunner, Kirkpatrick and Seidel [3]. They exclusively use geometry terminology to describe these shapes. The nerve definition is a by-product of the generalization of alpha shapes to three and higher dimensions [2, 4]. The idea of using the filter to compute connectivity is due to Delfinado and Edelsbrunner [1] who compute Betti numbers of alpha complexes in $\mathbb{R}^3$.

[1] C. J. A. Delfinado and H. Edelsbrunner. An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere. *Comput. Aided Geom. Design* **12** (1995), 771–784.

[2] H. Edelsbrunner. The union of balls and its dual shape. *Discrete Comput. Geom.* **13** (1995), 415–440.

[3] H. Edelsbrunner, D. G. Kirkpatrick and R. Seidel. On the shape of a set of points in the plane. *IEEE Trans. Inform. Theory* **IT-29** (1983), 551–559.

[4] H. Edelsbrunner and E. P. Mücke. Three-dimensional alpha shapes. *ACM Trans. Graphics* **13** (1994), 43–72.