Voronoi and Delaunay Complexes

Topics: Voronoi cells, Delaunay complex, distance maps, linearization, polarity, polar polyhedra.

Voronoi cells. In analogy to the two-dimensional case we define the *Voronoi cell* of a point p in a finite collection $S \subseteq \mathbb{R}^3$ as the set of points at least as close to p as to any other point in S:

$$V_p \quad = \quad \{x \in \mathbb{R}^3 \mid \|x - p\| \leq \|x - q\|, \ \forall q \in S\}.$$

Each inequality defines a closed half-space, and V_p is the intersection of a finite collection of such half-spaces. In other words, V_p is a convex polyhedron, maybe like the one shown in Figure 1. The probability that four or more planes meet in a common point is zero. It follows that in the generic case every vertex of V_p belongs to only three facets, and as a consequence it belongs to only three edges of the polyhedron. If V_p is bounded then it is the convex hull of its vertices. It is also possible that V_p is unbounded. This is the case iff there is a plane through p with all points of S on or on one side of the plane.

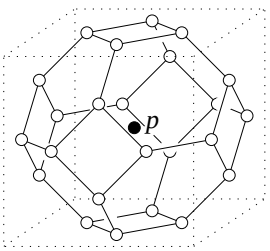

Figure 1: The Voronoi cell of a point in a body centered cube lattice. The relevant neighbors of the cube center p are the eight cube corners and the centers of the six adjacent cubes.

A point x that belongs to k Voronoi cells is equally far from the k generating points. It follows that the k points lie on a common sphere. If the points are in general position then $k \leq 4$. A Voronoi vertex x belongs to at least four Voronoi cells, and assuming general position it belongs to exactly four cells.

Delaunay complex. Same way as in two dimensions, we obtain the *Delaunay complex* by taking the dual of the Voronoi complex. The Delaunay vertices are the points in S. The Delaunay edges connect generators of Voronoi cells that share a common facet. The Delaunay facets connect generators of Voronoi cells that share a common edge. Assuming general position, each edge is shared by exactly three Voronoi cells, and in this case the Delaunay facets are triangles. The Delaunay cells connect generators of Voronoi cells that share a common vertex. Assuming general position, the number of such Voronoi cells is four and the Delaunay cells are tetrahedra. Consider point p in Figure 1. Its Voronoi cell has 14 facets, 36 edges, and 24 vertices. It follows that p belongs to 14 Delaunay edges, 36 Delaunay triangles, and 24 Delaunay tetrahedra, as illustrated in Figure 2.

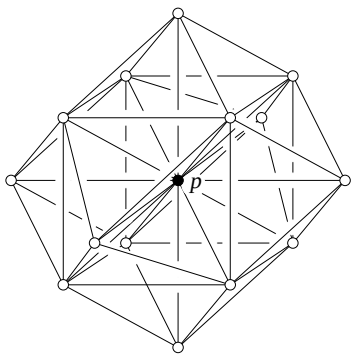

Figure 2: The Delaunay neighborhood of a cube center in a body centered cube lattice.

Assuming general position of the points in S, the Delaunay complex is a collection of simplices. To prove that it is a simplicial complex we still need to show the simplices avoid improper intersections. We do this by introducing geometric transformations that relate Voronoi and Delaunay complexes in $\mathbb{R}^3$ to boundary complexes of convex polyhedra in $\mathbb{R}^4$.

Distance maps. The square distance from $p \in S$ is the map $\pi_p : \mathbb{R}^3 \to \mathbb{R}$ defined by $\pi_p(x) = \|x - p\|^2$. Its graph is a paraboloid of revolution in $\mathbb{R}^4$. Figure 3 illustrates this idea in one lower dimension. Take

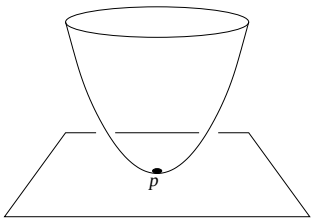

Figure 3: The graph of the square distance function of point p in the plane.

the collection of all square distance functions defined by points in S. The pointwise minimum is the map $\pi_S : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\pi_S(x) \quad = \quad \min\{\pi_p(x) \mid p \in S\}.$$

Its graph is the lower envelope of the collection of paraboloids. By definition of Voronoi cell, $\pi_S(x) = \pi_p(x)$ iff $x \in V_p$. We can therefore think of V_p as the projection of the portion of the lower envelope contributed by the paraboloid π_p.

Linearization. All square distance functions have a single quadratic term, which is $\|x\|^2$. If we subtract that term we get linear functions:

$$\begin{aligned} f_p(x) &= \pi_p(x) - \|x\|^2 \\ &= (x-p)^T \cdot (x-p) - x^T \cdot x \\ &= -2 \cdot p^T \cdot x + \|p\|^2. \end{aligned}$$

The graph of f_p is a hyperplane in $\mathbb{R}^4$. The same transformation warps the hyperplane $x_4 = 0$ to the upside-down paraboloid $x_4 = -\|x\|^2$. Figure 4 shows the result of the transformation applied to the plane and paraboloid in Figure 3. We can apply the transfor-

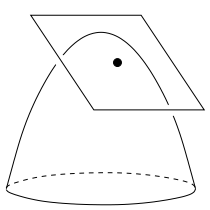

Figure 4: The plane in Figure 3 becomes an upside-down paraboloid, and the paraboloid becomes a plane.

mation to the entire collection of paraboloids at once. Each point in $\mathbb{R}^4$ travels parallel to the x_4-axis and the traveled distance is the square distance to that axis. Paraboloids go to hyperplanes, intersections of paraboloids go to intersections of hyperplanes, and the lower envelope of the paraboloids goes to the lower envelope of the hyperplanes.

Replace each hyperplane by the closed half-space bounded from above by the hyperplane. The intersection of the half-spaces is a convex polyhedron F in $\mathbb{R}^4$, and the lower envelope of the hyperplanes is the boundary of F. It is a complex of convex faces of dimension 3, 2, 1, 0. Note that the transformation moves points parallel to the x_4-axis. It follows that the projection to $x_4 = 0$ of the lower envelope of paraboloids and the lower envelope of hyperplanes are the same. In particular, the projection of each 3-dimensional face of F is a Voronoi cell, and the projection of the entire boundary complex of F is the Voronoi complex.

Polarity. We still have to describe what all this has to do with the Delaunay complex of S. Instead of addressing this question directly we first study the relationship between non-vertical hyperplanes and their polar points in $\mathbb{R}^4$.

A non-vertical hyperplane is the graph of a linear function $f : \mathbb{R}^3 \to \mathbb{R}$, which can generally be defined by a point $p \in \mathbb{R}^3$ and a scalar $c \in \mathbb{R}$:

$$f(x) \quad = \quad -2 \cdot p^T \cdot x + \|p\|^2 + c.$$

It will be convenient to ignore the difference between a linear function and its graph or hyperplane. The *polar point* of f is $g = f^* = (p, -\|p\|^2 - c)$. The hyperplane comes from a square distance function iff $c = 0$, and $c = 0$ iff the polar point lies on the upside-down paraboloid which is the image of $x_4 = 0$. Figure 5 shows a few examples of hyperplanes and their polar points in $\mathbb{R}^2$. Since the hyperplanes are non-vertical, we

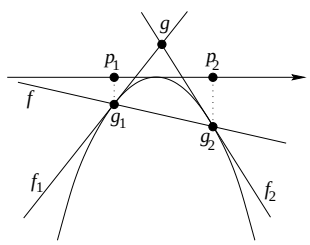

Figure 5: Points g_1, g_2, g are polar to the lines (hyperplanes) f_1, f_2, f. Lines f_1, f_2 are warped images of the distance square functions of the points p_1, p_2 on the real line.

can unambiguously define what it means that a point lies *above*, *on*, *below* a hyperplane. Let f_1, f_2 be two non-vertical hyperplanes and g_1, g_2 their polar points.

CLAIM 1. Point g_1 lies above, on, below hyperplane f_2 iff point g_2 lies above, on, below hyperplane f_1.

PROOF. Let $g_i = (p_i, -\|p_i\|^2 - c_i)$ for $i = 1, 2$. The algebraic expression for g_1 above f_2 is

$$-\|p_1\|^2 - c_1 \quad > \quad -2 \cdot p_2^T \cdot p_1 + \|p_2\|^2 + c_2.$$

We move terms left and right and use the fact that vector products are commutative to get

$$-\|p_2\|^2 - c_2 \quad > \quad -2 \cdot p_1^T \cdot p_2 + \|p_1\|^2 + c_1.$$

This is the algebraic expression for g_2 above f_1. The arguments for point g_1 lying on and below hyperplane f_2 are similar. □

Polar polyhedron. We are now ready to construct the Delaunay complex as the projection of the boundary complex of a convex polyhedron in $\mathbb{R}^4$. For each point $p \in S$ let $g_p = (p, -\|p\|^2)$ be the polar point of the corresponding hyperplane. All points g_p lie on the upside-down paraboloid, as shown in Figure 6. For a

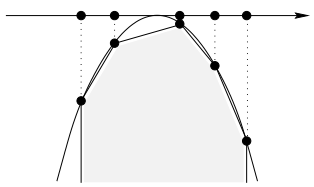

Figure 6: The boundary complex of the shaded polyhedron projects to the Delaunay complex of the points on the horizontal line.

non-vertical hyperplane f we consider the closed half-space bounded from above by f. Let G be the intersection of all such half-spaces that contain all points g_p. G is a convex polyhedron in $\mathbb{R}^4$. Its boundary consists of the upper portion of the convex hull boundary plus the silhouette extended to infinity in the $-x_4$ direction. Claim 1 implies the following correspondence between G and the convex polyhedron F. A hyperplane *supports* F if it has non-empty intersection with the boundary and empty intersection with the interior.

CLAIM 2. A hyperplane f supports F iff the polar point $g = f^*$ lies in the boundary of G.

Imagine exploring F by rolling the supporting hyperplane along its boundary. The dual image of this picture is the polar point moving around in the boundary of G. For each k-dimensional face of F we get a $(3-k)$-dimensional face of G and vice versa. An exception are the vertical faces of G, which do not correspond to any faces of F except possibly to faces stipulated at infinity. The relationship between the two boundary complexes is the same as that between the Voronoi and the Delaunay complexes. The isomorphism between the boundary complex of F and the Voronoi complex implies the isomorphism between the boundary complex of G (excluding vertical faces) and the Delaunay complex. Since the vertices of G project to points in S it follows that the boundary complex of G projects to the Delaunay complex of S. This finally implies that there are no improper intersections between Delaunay simplices. The Delaunay complex of a set S of finitely many points in general position is indeed a simplicial complex.

Bibliographic notes. Voronoi diagrams and Delaunay triangulations or complexes are named after Georges Voronoi [3] and Boris Delaunay (also Delone) [1]. The concepts themselves are older and can be traced back to prominent mathematicians of earlier centuries, including Friedrich Gauß and René Descartes. Also the connection to convex polytopes has been known for a long time. The combinatorial theory of convex polytopes is a well developed field within mathematics. We refer to the texts by Branko Grünbaum [2] and by Günter Ziegler [4] for excellent sources of the accumulated knowledge in that subject.

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[4] G. M. ZIEGLER. *Lectures on Polytopes.* Springer-Verlag, New York, 1995.